A Note on Buyers' Behavior in Auctions with an Outside Option

Alexander Maslov*

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Abstract

In this note I show that the equilibrium in cutoff strategies observed in auctions with a buy-it-now price may also arise in markets where objects are sold simultaneously by auctions and posted prices. However, contrary to auctions with a buy-it-now price where buyers need to know only the total number of players in the market, in the latter environment buyers must also observe the number of active bidders in the auction for the equilibrium to exist in cutoff strategies.

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JEL Codes: C70, D44

1 Introduction

Last decade has witnessed rampant development of e-commerce. Today, there are several online platforms (primarily in the consumer-to-consumer segment), which allow sellers to list their goods either directly by posted prices or auctions. eBay is likely the most popular consumer-to-consumer platform. When there is an effectively infinite number of homogeneous objects listed by posted prices and auctions, buyers have to choose whether they want to bid in the auction or purchase the same good from the posted price. The major parameters of decision are their valuation for the object and

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*Department of Economics, Finance and Quantitative Analysis, Kennesaw State University, 560 Parliament Garden Way, Kennesaw, GA, 30144, e-mail: alexander.maslov@kennesaw.edu

time of arrival. If a buyer places a high value on the object or arrives to the market when the auction has just commenced, he is more likely to buy the good at the posted price rather than bid in the auction. The literature on simultaneous use of auctions and posted prices dealt with this situation and showed that in the equilibrium buyers follow a threshold strategy (Etzion et al. (2006); Caldentey and Vulcano (2007); Sun (2008); Etzion and Moore (2013); Hummel (2015)). Two main assumptions were made: buyers committed to the chosen mechanism, and the number of goods was infinitely large.

In this note I consider a case with limited supply of objects and buyers who can bid dynamically switching between the mechanisms. In contrast to the situation with unlimited supply and "locked-in" buyers, some of them may not be able to purchase the good at all. It creates an incentive for buyers who can afford the object outside to exit the auction before its price matches the price of the good outside. These incentives turn out to be similar to the ones observed in auctions with a buy-it-now (BIN) price, but the conditions for them to exist are different.

In auctions with a BIN price, bidders with high valuations have an incentive to end the auction early by buying the good at the BIN price (Reynolds and Wooders (2009)). This mechanism has received much scrutiny, e.g. Mathews and Katzman (2006), Anderson et al. (2008), Reynolds and Wooders (2009), Chen et al. (2013), Anwar and Zheng (2015). The focus of the literature was on sellers' profits and rationale for using the BIN price. Here, I do not attempt to characterize sellers' equilibrium on markets with auctions and posted prices where the supply is limited.[1] The focus of this note is to analyze the differences between auctions with a BIN price and auctions with an outside option from the buyers' perspective. I show that when the buyout price is posted outside the auction, buyers follow cutoff strategies only when they are able to observe the number of active bidders. As shown by Reynolds and Wooders (2009), in auctions with a BIN price, this information is not required for buyers to follow cutoff strategies. In other words, in an environment with both an auction and a posted price the necessary condition is that buyers need to not only know the total number of buyers, but also the number of active buyers in the auction for a cutoff strategy to exist.

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[1] This attempt has been done in another paper by Maslov (2018).

2 The Model

Consider a market where two homogeneous goods are put up for sale. One of them is offered by an ascending clock auction, the other one is offered at a posted price p. To keep things simple I assume that there are three risk neutral buyers with valuations v_i who have unitary demands and independent private values distributed according to a cumulative distribution function $F(\cdot)$ with support $[\underline{v}, \overline{v}]$, where $f(\cdot)$ is continuous and positive with $\underline{v} \geq 0$ and $\underline{v} < p < \overline{v}$. Reservation price is normalized to zero.[2]

In the beginning, all buyers decide whether they want to buy the good at the posted price or start bidding in the auction, but have a choice to exit it later at any time and buy the good at the posted price. The auction ends when only one buyer remains. This buyer receives the object and pays the price at the clock. A buyer who decides to exit the auction and buy the good at the posted price gets the good and pays its price p. If several bidders attempt to exit the auction and buy the good at the posted price at the same time, the good is awarded randomly to one of the buyers, and the rest return to the auction.[3]

Clearly, buyers with valuations less than the posted price outside cannot afford it and thus bid their valuations (Vickrey (1961)). They drop out from the auction when the clock moves past their valuations. Buyers with valuations higher than the posted price never have an incentive to drop out from the auction. On the other hand, bidding their valuations cannot be best response as well. To see this, consider three buyers with $v_i > p$ bidding in the auction. When the clock reaches p each of them would want to exit and buy the good at p, but only one is randomly permitted to do so. The remaining buyers continue bidding in the auction and eventually receive a lower payoff from either having to pay more (since $v_i > p$) or losing the auction. Any buyer can do better by exiting the auction right before the clock matches the price outside. However, other buyers forseeing this may exit even earlier. Hence, there is such a price $c \in [\underline{v}, p]$ at which a buyer exits the auction, and there is no loss of generality in looking for an equilibrium in cutoff strategies when valuations are higher than the posted price outside.

A strategy for a bidder with a valuation higher than the posted price, who comes

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[2] The model could be generalized to n risk-averse buyers and include a reserve price, but it does not affect the intuition based on a technically simpler framework used in this paper.

[3] The following allocation rule may potentially create discontinuity (also well documented in auctions with a BIN price) in the cutoff function, because some buyers can do better by exiting at the start of the auction rather than bidding in the auction where everyone follows symmetric strategy.

to the market (at the start of the auction) is a function $z(\cdot)$, which gives each $v \in (p, \overline{v}]$ a cutoff bid price $c \in [0, p)$ at which the bidder leaves the auction and buys the good at p. The strategy is defined similarly to auctions with a BIN price. However, if a buyer is allowed to observe the number of buyers, then at price c he is able to update the information on the number of active buyers in the auction and revise his strategy.

3 Comparing cutoff strategies

Reynolds and Wooders (2009) proved that the equilibrium in cutoff strategies exists in auctions with a BIN price when $E(v - x|c \leq x \leq v) < v - B$, where x is the first order statistic of all buyers except the buyer with valuation v, and B is the buyout price. In auctions with a BIN price, the expected payoff of a buyer depends only on the realization of the first order statistic among his opponents, because the former buyer's beliefs about the latter valuation are updated directly via the clock.[4] In other words, in a BIN setting, if the auction is active, it means that the clock has not yet passed the first order statistic, and a buyer does not need to know the exact number of active bidders. In contrast, when there is an additional object outside the auction (instead of the BIN price), the expected payoff of a buyer also depends on the realization of the second order statistic among his opponents. Whether the auction is active or not does not provide any information on the second order statistic. To see this, consider a buyer who follows a cutoff strategy. In a BIN setting, if the auction ended before the clock reached the price at which this buyer was supposed to use the BIN option, it means that the first order statistic was below this price. In a setting with an outside price, when the clock reaches the exit price of the buyer, if the second order statistic has dropped out, it is no longer profitable to follow a cutoff strategy, because there are only two buyers left in the market, and each of them may get the good at most at p. However, this information may only be received from observing the second order statistic.

First, I allow buyers to observe the number of active bidders in the auction. Consider a buyer with a value v. Let his opponents have valuations x and y with $x > y$. In other words, x is the first (highest) order statistic and y is the second order statistic. Then, their joint density is equal to $g(x, y) = 2f(x)f(y)$. To calculate the expected

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[4] If $x < c$, then x drops out of the auction, and it ends before the clock reaches c. If $x > v$ then the buyout option is used before c, and the auction again ends before the clock reaches c.

payoff of the buyer with valuation v we need to consider all possible realizations of his opponents' values (Figure 1 illustrates this example).

$$
\begin{aligned}
\max_{c} \Bigg(& \int_{\underline{v}}^{c} \int_{\underline{v}}^{x} (v-x)\Omega + \int_{c}^{p} \int_{\underline{v}}^{c} (v-x)\Omega + \int_{c}^{p} \int_{c}^{x} (v-p)\Omega \\
& + \int_{p}^{z^{-1}(c)} \int_{\underline{v}}^{x} (v-p)\Omega + \int_{z^{-1}(c)}^{v} \int_{\underline{v}}^{z(x)} (v-p)\Omega + \int_{z^{-1}(c)}^{v} \int_{z(x)}^{x} (v-y)\Omega \\
& + \int_{v}^{\overline{v}} \int_{\underline{v}}^{z(x)} (v-p)\Omega + \int_{v}^{\overline{v}} \int_{z(x)}^{v} (v-y)\Omega \Bigg)
\end{aligned}
\tag{1}
$$

where $\Omega = 2f(x)f(y)dydx$.

The first two terms are the payoffs of the buyer with valuation v when y drops out before c. It is easy to see that in this case the former buyer no longer exits at c, but bids $max[v, p]$ instead. The next two terms are the payoffs from following the cutoff strategy. The last terms are the payoffs of the buyer when his valuation is lower than the first order statistic among his opponents conditional on whether the second order statistic is below the cutoff price of the first order statistic or above it.

When a buyer with valuation v is able to observe the number of active bidders, at c, he will continue following his strategy if the number of buyers remained the same. As stated above, if one buyer dropped out from the auction before c, then it is the best response of the buyer with valuation v to bid $max[v, p]$. If one of his opponents left the auction before c and bought the good outside, then the buyer with valuation v simply bids his valuation. Notice that the revision of the strategy is conditional on the observance of the buyer with valuation y (second order statistic among his opponents) when the clock reaches c.

Simplifying the expression above produces the following slope field (see Appendix for details):

$$
z'(v) = \frac{f(v)}{f(c)} \frac{p(F(v) - F(c)) - \int_c^v y f(y) dy}{F(c)(c-p) + \int_c^p F(x) dx}
\tag{2}
$$

For the equilibrium in cutoff strategies to exist, the derivative of the cutoff function needs to be negative. Observe that the denominator of (2) is positive, because $F(c)(c-p) + \int_c^p F(x)dx > F(c)(c-p) + \int_c^p F(c)dx = 0$. Hence, for the whole fraction to be negative the numerator should also be negative which happens only when

$p(F(v)-F(c)) < \int_c^v yf(y)dy$ or $p < \frac{\int_c^v yf(y)dy}{F(v)-F(c)}$, where the latter term is the expected value of the buyer with valuation y on the interval $[c, v]$. In other words, a buyer with valuation v follows a cutoff strategy only if the valuation of the second order statistic among his opponents is higher than the posted price outside, i.e. $E(v-y|z(v) \leq y \leq v) < v-p$.[5]

The result above is intuitive. If all buyers are bidding in the auction, and at price c the number of active bidders did not change, a buyer with valuation v expects the second order statistic among his opponents to be higher than p and follows a cutoff strategy. If the lowest-valuation buyer has dropped out before c, the former buyer is able to observe it and no longer follows a cutoff strategy, but instead bids $max[v, p]$.

Now, consider the objective function of a buyer with valuation v who can no longer observe the number of active bidders. There are two main alterations. If y drops out before c, a buyer with valuation v does not observe it and follows cutoff strategy $z(\cdot)$ earning $v-p$ instead of $v-x$. Additionally, when $x > v$ and $y < z(x)$, the buyer with valuation x does not observe that y has dropped out before his cutoff price and follows the cutoff strategy. The buyer with valuation v now wins the auction and pays $v-z(x)$ instead of $v-p$:

$$\begin{aligned}\max_c \Big(&\int_{\underline{v}}^{c}\int_{\underline{v}}^{x}(v-x)\Omega + \int_{c}^{z^{-1}(c)}\int_{\underline{v}}^{x}(v-p)\Omega + \int_{z^{-1}(c)}^{v}\int_{\underline{v}}^{z(x)}(v-z(x))\Omega \\ &+ \int_{z^{-1}(c)}^{v}\int_{z(x)}^{x}(v-y)\Omega + \int_{v}^{\overline{v}}\int_{\underline{v}}^{z(x)}(v-z(x))\Omega + \int_{v}^{\overline{v}}\int_{z(x)}^{v}(v-y)\Omega \Big)\end{aligned} \tag{3}$$

where $\Omega = 2f(x)f(y)dydx$.

Simplifying the expression above by an analogy with 1 produces:

$$z'(v) = \frac{f(v)}{f(c)}\frac{pF(v) - cF(c) - \int_c^v yf(y)dy}{F(c)(p-c)} \tag{4}$$

The denominator of equation 4 is positive, but the numerator may be either positive or negative. Unlike the previous example, where there was a strict condition under which it is negative, in this case, it can be negative conditional on a potentially infinite combination of the parameters. Hence, it is impossible to define when exactly equilibrium in cutoff strategies will exist.

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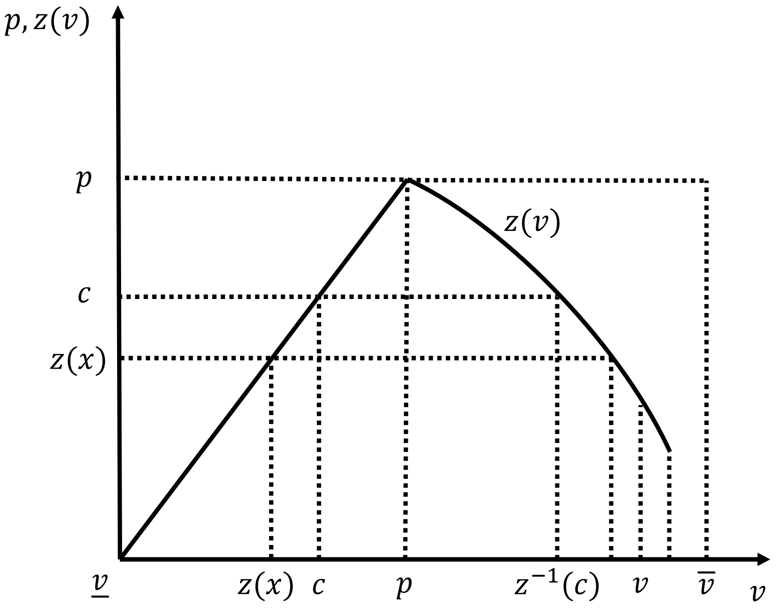
[5] Notice that this condition is identical to the one in auctions with a BIN price, except that now it is based on second order statistic rather than the first order statistic.

The intuition for the above result is as follows. If a buyer is not able to observe the number of bidders in the auction, when the clock reaches his cutoff price, he has to follow a cutoff strategy unless the clock stopped beforehand, meaning that the valuation of the first order statistic among his opponents was lower than c. If the second order statistic has dropped out from the auction before c, then it is no longer a payoff-maximizing strategy to exit the auction. Hence, when a buyer is not able to observe the number of active bidders, in some cases (when $y > c$) he would be better off following a cutoff strategy, while in other cases (when $y < c$) he would be better off changing his strategy to bidding $max[v, p]$.

4 Conclusion

In this note I have compared buyers' cutoff strategies in auctions with a BIN price and auctions with an outside price. Unlike auctions with a BIN price, in an environment with both an auction and posted price outside the necessary condition is that buyers need to not only know the total number of buyers, but also the number of active bidders in the auction for a cutoff strategy to exist. Without being able to revise a strategy conditional on the information obtained from the second highest order statistic, following cutoff strategies is no longer payoff-maximizing. In principle, a buyer only needs to observe the second-highest order statistic among his opponents to follow cutoff strategies. However, in reality, to receive information specifically on the second-order statistic means that a buyer needs to observe all active bidders in the auction.

The intuition behind the model developed in this note may be extended to multiple objects outside of the auction. The presence of each additional object outside will require a buyer to observe a lower order statistic. For example, when there is an auction with three objects listed at posted prices outside, a buyer will need to observe if the third highest order statistic among his opponents dropped out from the auction before the clock reached his cutoff price.



$x \in [\underline{v}, c]; y \in [\underline{v}, x]$	$(v - x)$
$x \in [c, p]; y \in [\underline{v}, c]$	$(v - x)$
$x \in [c, p]; y \in [c, x]$	$(v - p)$
$x \in [p, z^{-1}(c)]; y \in [\underline{v}, x]$	$(v - p)$
$x \in [z^{-1}(c), v]; y \in [\underline{v}, z(x)]$	$(v - p)$
$x \in [z^{-1}(c), v]; y \in [z(x), x]$	$(v - y)$
$x \in [v, \overline{v}]; y \in [\underline{v}, z(x)]$	$(v - p)$
$x \in [v, \overline{v}]; y \in [z(x), v]$	$(v - y)$

Figure 1: Equilibrium bidding functions.[6]

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[6] Note that $z^{-1}(c)$ may be to the left or right of the "jump-down". The same pertains to x and y when $x, y > z^{-1}(c)$. Also, it is easy to show that when $p < v < z^{-1}(c)$ the domains of integration are unchanged, i.e. $\int_p^v \int_{\underline{v}}^x (v-p)2f(x)f(y) + \int_v^{z^{-1}(c)} \int_{\underline{v}}^x (v-p)2f(x)f(y)dydx = \int_p^{z^{-1}(c)} \int_{\underline{v}}^x (v-p)2f(x)f(y)dydx$. However, when $z^{-1}(c) < v < \overline{v}$ domains of integration change conditional on whether $v > x$ or $x > v$ (the figure shows the case when v is greater than x).

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A The derivative of the cutoff function

To begin, note that all parts of equation (1) may be divided by 2. Applying the Leibniz Rule to each of the integrals separately produces[7]:

$$\begin{gathered}\frac{d}{dc}\int_{\underline{v}}^{c}\int_{\underline{v}}^{x}(v-x)f(x)f(y)dydx=\frac{d}{dc}\int_{\underline{v}}^{c}(v-x)f(x)\int_{\underline{v}}^{x}f(y)dydx=\\ \frac{d}{dc}\int_{\underline{v}}^{c}(v-x)f(x)F(x)dx=(v-c)f(c)F(c)\end{gathered}\tag{5}$$

$$\begin{gathered}\frac{d}{dc}\int_{c}^{p}\int_{\underline{v}}^{c}(v-x)f(x)f(y)dydx=\frac{d}{dc}F(c)\int_{c}^{p}(v-x)f(x)dx=\\ f(c)\int_{c}^{p}(v-x)f(x)dx-F(c)f(c)(v-c)=\\ f(c)\Big((v-p)F(p)-(v-c)F(c)+\int_{c}^{p}F(x)dx-F(c)(v-c)\Big)=\\ f(c)\Big((v-p)F(p)-2(v-c)F(c)+\int_{c}^{p}F(x)dx\Big)\end{gathered}\tag{6}$$

$$\begin{gathered}\frac{d}{dc}\int_{c}^{p}\int_{c}^{x}(v-p)f(x)f(y)dydx=(v-p)\frac{d}{dc}\int_{c}^{p}f(x)(F(x)-F(c))dx=\\ (v-p)\frac{d}{dc}\Big(\int_{c}^{p}f(x)F(x)dx-F(c)\int_{c}^{p}f(x)dx\Big)=\\ (v-p)\Big(-f(c)F(c)-\frac{d}{dc}F(c)(F(p)-F(c))\Big)=\\ (v-p)\Big(-f(c)F(c)-f(c)(F(p)-F(c))+f(c)F(c)\Big)=\\ -(v-p)f(c)(F(p-F(c))\end{gathered}\tag{7}$$

$$\begin{gathered}\frac{d}{dc}\int_{p}^{z^{-1}(c)}\int_{\underline{v}}^{x}(v-p)f(x)f(y)dydx=\frac{d}{dc}\int_{p}^{z^{-1}(c)}(v-p)f(x)F(x)dx=\\ \frac{(v-p)f(z^{-1}(c))F(z^{-1}(c))}{z'(z^{-1}(c))}\end{gathered}\tag{8}$$

$$\begin{gathered}\frac{d}{dc}\int_{z^{-1}(c)}^{v}\int_{\underline{v}}^{z(x)}(v-p)f(x)f(y)dydx=\frac{d}{dc}\int_{z^{-1}(c)}^{v}(v-p)f(x)F(z(x))dx=\\ -\frac{(v-p)f(z^{-1}(c))F(c)}{z'(z^{-1}(c))}\end{gathered}\tag{9}$$

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[7] Note that $F(\underline{v})=0$, $F(\overline{v})=1$ and $\frac{d}{dc}z^{-1}(c)=\frac{1}{z'(z^{-1}(c))}$.

$$
\begin{aligned}
\frac{d}{dc}\int_{z^{-1}(c)}^{v}\int_{z(x)}^{x}(v-y)f(x)f(y)dydx = -\frac{f(z^{-1}(c)}{z'(z^{-1}(c))}\int_{c}^{z^{-1}(c)}(v-y)f(y)dy+ \\
\int_{z^{-1}(c)}^{v}\frac{d}{dc}\left[f(x)\int_{z(x)}^{x}(v-y)f(y)dy\right]dx
\end{aligned}
\tag{10}
$$

Notice that $\frac{d}{dc}\left[f(x)\int_{z(x)}^{x}(v-y)f(y)dy\right] = 0$. Hence, integrating by parts $\int_{c}^{z^{-1}(c)}(v-y)f(y)dy$ and rearranging gives:

$$
-\frac{f(z^{-1}(c)}{z'(z^{-1}(c))}\Big[(v-z^{-1}(c))F(z^{-1}(c))-(v-c)F(c)+\int_{c}^{z^{-1}(c)}F(y)dy\Big]
\tag{11}
$$

Finally, both $\frac{d}{dc}\int_{v}^{\overline{v}}\int_{\underline{v}}^{z(x)}(v-p)2f(x)f(y)dydx$ and $\frac{d}{dc}\int_{v}^{\overline{v}}\int_{z(x)}^{v}(v-y)2f(x)f(y)dydx$ are equal to 0, because both integrals do not depend on c.

In equilibrium $z^{-1}(c)=v$ and $c=z(v)$. Hence, the elements with $v-z^{-1}(c)$ are equal to 0. Substituting each integral back into the original equation and rearranging gives:

$$
\begin{gathered}
f(c)\Big(F(c)(c-p)+\int_{c}^{p}F(x)dx\Big) = \\
\frac{f(v)\Big((v-p)(F(c)-F(v))-(v-c)F(c)+\int_{c}^{v}F(y)dy\Big)}{z'(v)}
\end{gathered}
\tag{12}
$$

Expressing the derivative and further simplifying the numerator results in:

$$
z'(v)=\frac{f(v)}{f(c)}\frac{F(c)(c-p)-F(v)(v-p)+\int_{c}^{v}F(y)dy}{F(c)(c-p)+\int_{c}^{p}F(x)dx}
\tag{13}
$$

Finally, integrating by parts $\int_{c}^{v}1\times F(y)dy$, the numerator can be rewritten as $F(c)(c-p)-F(v)(v-p)+F(v)v-F(c)c-\int_{c}^{v}yf(y)dy$ or equivalently $p(F(v)-F(c))-\int_{c}^{v}yf(y)dy$, which produces the final equation:

$$
z'(v)=\frac{f(v)}{f(c)}\frac{p(F(v)-F(c))-\int_{c}^{v}yf(y)dy}{F(c)(c-p)+\int_{c}^{p}F(x)dx}
\tag{14}
$$