

## **Joint pricing and inventory decisions in competitive retail**

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## Abstract

In this paper, we evaluate the joint pricing and inventory decisions for retailers in a competitive market. Specifically we develop a demand models for the optimization of retail sales of perishable products as a differential game, solving the joint inventory and pricing problem over a finite horizon. We examine general pricing policies over the sales horizon for both symmetric and asymmetric firms within the market and discuss the implications of both. We examine specific strategies for the smaller firm, such as service differentiation, when competing with a much larger competitor in an asymmetric duopoly, as well as the role of price leadership as a strategy for the larger firm when the market is highly competitive.

*Keywords:* Differential games, Optimal pricing, Joint pricing & inventory, Retail

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## 1. Introduction

An obvious problem common to the retail industry is the estimation of the number of units of a product to be stocked and the price that the retailer should charge in order to maximize profit. One avenue available to address these issues is the application of revenue management, which was originally developed for pricing of perishable products, particularly in the

airline industry. As such, the retail sector has seen a rapid expansion in the last several years of research focussed on revenue management and related tools. Recent publications involving applications to the retail sector include [1–8]. These practices have been commonly applied to perishable products, i.e. those that have a short shelf life as a result of product aging and physical deterioration, or where product value is decreasing over time, such as fashion or high-technology goods [9], and thus fit the revenue management framework. Recent review articles by Elmaghraby and Keskinocak [10] and Bitran and Caldentey [11], as well as the text by Talluri and van Ryzin [12] provide a comprehensive treatment of the current pricing and revenue literature.

There are two primary areas of research being considered, that of the joint pricing-inventory problem as well as competition, specifically in the retail environment. The lack of modeling within the pricing and revenue management field in general with respect to the competitive environment has been highlighted within the review literature [10, 11]. Within the joint pricing-inventory literature in particular, there is often a focus on the monopolist problem. This research is typically based upon a newsvendor-type problem, for example [13–15], though others have looked at different aspects including alternative markets such as internet retailing [16] or inclusion of various additional factors such as ordering costs, back-logging, and lost sales [17, 18]. Other extensions include pricing and inventory with multiple products [19–21] for a single firm. However, studies have shown the dominant industry structure is that of an oligopoly [22] involving imperfect competition, thus enabling the price discrimination observed in the marketplace [23]. In addition, with respect to industry pricing practices, recent empiri-

cal analysis [24] indicate that competitor pricing was the single most critical determinant of retail pricing strategies. Of the retail literature listed above, Rajan and Steinberg [1], Smith and Achabal [4] and Federgruen and Heching [5] further addressed joint pricing and inventory decisions. However, still much needs to be done. In particular, though various research studies have looked at the joint pricing-inventory problem, and dynamic pricing under competition, there are very few papers that have done both.

A review of the recent joint pricing-inventory literature can be found in [25] as well as [26]. Thus, here we highlight papers that focus on competitive retail environments in which both pricing and inventory decisions are considered. Two recent papers that evaluate joint price-inventory problems in a retail environment are Adida and Perakis [27] and Bernstein and Federgruen [28]. In [27] the authors study a make to stock production system involving two firms and evaluate dynamic pricing and inventory decisions, while [28] develop models for oligopolies with competing retailers under demand uncertainty over an infinite-horizon.

One approach to modeling competition is through differential games [29, 30], which provides for a dynamic competition formulation, while employing the analysis tools available from optimal control and differential equations. Specifically we propose a continuous-time differential game model of a competitive retail market where the firms may be similar (symmetric) or where a smaller retailer competes with a much larger firm such as a big-box retailer (asymmetric). We solve for both the equilibrium prices as well as the optimal inventory stocking levels, and determine the resulting profits for both firms. Further, we develop practical bounds on the solutions when the firms

are asymmetric, and discuss the implications of modeling in such a market. Specific pricing and inventory strategies are determined for both firms as a function of market conditions. The remainder of the paper is organized as follows. In Section 2 we provide a description of the competitive demand model. In Section 3 we define the differential game, and examine requirements for an equilibrium solution as well as properties of the solution. Numeric results are presented in Section 4, including insights into pricing and inventory policies. Section 5 offers conclusions as well as extensions for future research.

## 2. Model Formulation

We develop a model of a competitive retail environment, where products have perishable value, such as seasonal goods due to a limited sales horizon or high-tech products in their initial introductory sales period. For example, the above would be true of consumer electronics such as laptop computers, which see rapid advances in product offerings. These products often have a short retail sales life, are durable in nature with respect to repeat purchase over the period, and involving consumers knowledgeable of the market and product values. Retailers can compete on price as well as other dimensions. Specifically, we allow for direct price interactions between the firms, while allowing for differences between individual retailers and how this impacts on market share, and thus inventory and pricing decisions. Table 1 provides a summary of model notation. Throughout, subscripts on the parameters indicate firm specific values where appropriate.

We propose demand capture or market share as the state variable for individual firms, with an overall market,  $M(p)$ , that is price sensitive. This

Table 1: Model Notation

Parameter	Description
$P(t)$	Price per unit inventory at time $t$
$\bar{P}(t)$	Specified the markets average upper bound price at time $t$
$M(t)$	Total market size at time $t$
$X(t)$	Demand capture/market share at time $t$ ; total for all firms is 1
$I(t)$	Inventory level at time $t$
$Q(t)$	Cumulative sales to time $t$
$\phi$	Scaling parameter for price differential of market share
$\alpha$	Rate parameter in demand models
$\gamma$	Scaling parameter for market size
$C$	Cost per unit inventory
$S$	Salvage value per unit inventory at end of the sales horizon
$\eta$	Residual value of market share at end of the sales horizon
$T$	Length of sales horizon
$\rho$	Discount rate
$\eta_i$	Discounted future or salvage value of market share
$N$	Number of firms

market share specification is similar to the Lanchester model commonly employed in advertising games (for example see [31]). Specifically, with a total market involving  $N$  firms, the general formulation for market share of firm 'i' is:

$$dX_i(t) = \alpha \left( \sum_{j \neq i} \frac{\phi_j P_j(t)}{N-1} - \phi_i P_i(t) \right) dt \quad (1)$$

with

$$\sum_{i=1}^N X_i(t) = 1, \quad X_i(t) \geq 0 \quad \forall i, \quad t \in [0, T] \quad (2)$$

The rate parameter,  $\alpha$ , gives the market's response to the firms prices, and reflects how quickly the market becomes aware of and/or reacts to a price differential. The weighting ( $\phi$ ) on individual prices in the price differential

reflects the ability to have different prices in the market without a change in the market share of individual firms. This can be interpreted as firm preference based upon service, location, etc. for firms selling identical goods, or a different consumer valuation for substitutable products.

The size of the market available is taken as price-sensitive, with the market as a function of average firm prices:

$$M(t) = \gamma \left( \bar{P}(t) - \frac{\sum_{i=1}^N P_i(t)}{N} \right) \quad i = 1, \dots, N \quad (3)$$

where  $\frac{\sum_{i=1}^N P_i(t)}{N}$  is the average price for the  $N$  firms in the market. This is similar to the approach adopted by Rao [32] for a price sensitive market potential. Specifically, the market is taken as linear and downward sloping in the firms' prices, and essentially gives the response to the market's willingness to pay.  $\bar{P}(t)$  represents the maximum averaged prices the market for the firm is willing to accept for the product, while  $\gamma$  simply scales the relative market size.  $\bar{P}(t)$  is time-dependent, reflecting the changing value consumers may place on the item over time. For example, with perishable value items such as fashion goods or high technology products, we anticipate that  $\bar{P}(t)$  will decrease over the sales horizon. It is assumed that each firm can be differentiated along either product or other attributes as discussed above. For this reason, the effect of the firm's price on the market is taken as a function of  $\phi$ .

Each firm attempts to maximize overall profits, resulting in the objective function for a firm 'i' over a finite horizon (T):

$$V_i^*(M, X, I, t; P_i) = \max_{P_i(\tau) \in \mathcal{U}_i} \left\{ \int_0^T e^{-\rho\tau} P_i(\tau) M(\tau) X_i(\tau) d\tau - C_i I_i(0) + e^{-\rho T} S_i I_i(T) + \eta_i X_i(T) \right\} \quad (4)$$

$\mathcal{U}_i$  is the domain or set of prices for firm  $i$ . Inventory at time  $T$  is sold for salvage for amount  $S_i$ , with  $S_i < C_i$  to prevent initially stocking units that can be sold for more than cost at the end of the sales horizon. The final term  $\eta_i$  represents the estimated discounted long-term value of market share at the end of the horizon for the specific product, and is similar to the approach described in [33]. This could be interpreted, for example, as the value of future sales of similar products as a result of a repeat customers, with  $\eta(T) \geq 0$ . Finally, for joint inventory/pricing over a finite horizon, the change in cumulative sales (Q), or change in inventory, is expressed as:

$$dQ_i(t) = -dI_i(t) = M(t)X_i(t)dt \quad (5)$$

### 3. Solutions of the Differential Game

For the market share model described above, we specify the form the solution equations take and describe pricing and inventory policy implications for each. As our problem of interest is for perishable asset value goods over a short finite horizon, we consider the discounting term  $\rho$  to be negligible throughout the remainder of the paper. For the total market formulation with demand capture as defined in (1), the resulting Hamiltonian for firm 'i' is defined as:

$$H_i = (P_i(t) - C_i)M(t)X_i(t) + \lambda_{i,i}(t) \alpha \left( \sum_{j \neq i} \frac{\phi_j P_j(t)}{N-1} - \phi_i P_i(t) \right) + \sum_{j \neq i} \lambda_{i,j}(t) \alpha \left( \sum_{k \neq j} \frac{\phi_k P_k(t)}{N-1} - \phi_j P_j(t) \right) \quad (6)$$

where  $\lambda_{i,i}(t)$  and  $\lambda_{i,j}(t)$  are the associated adjoint variables. Evaluating the first-order condition (FOC) for (6), we get

$$\frac{\partial H_i}{\partial P_i(t)} = \gamma \left( \bar{P}(t) - \sum_i \frac{P_i(t)}{N} \right) X_i(t) - \frac{(P_i(t) - C_i)\gamma X_i(t)}{N} - \lambda_{i,i}(t)\alpha\phi_i + \sum_{j \neq i} \lambda_{i,j}(t) \frac{\alpha\phi_i}{N-1} = 0 \quad (7)$$

which results in the candidate solution

$$P_i^*(t) = \frac{N\bar{P}(t) + C_i - \sum_{j \neq i} P_j(t)}{2} - \frac{N\alpha\phi_i}{2\gamma X_i(t)} \left( \lambda_{i,i}(t) - \frac{\sum_{j \neq i} \lambda_{i,j}(t)}{N-1} \right) \quad (8)$$

For the open loop solution, the adjoint variables must satisfy the following differential equations:

$$\dot{\lambda}_{i,i}(t) = -\partial H_i / \partial X_i(t) = -(P_i(t) - C_i)\gamma \left( \bar{P}(t) - \frac{\sum_i P_i(t)}{N} \right) \quad (9)$$

and

$$\dot{\lambda}_{i,j}(t) = -\partial H_i / \partial X_j(t) = 0 \quad (10)$$

along with transversality conditions  $\lambda_{i,i}(T) = \eta_i$  and  $\lambda_{i,j}(T) = 0$ , for  $i, j = 1, \dots, N$ ;  $i \neq j$ . Thus from (10) and the transversality conditions, it is obvious that  $\lambda_{i,j}(t) = 0$ ,  $\forall j$  and  $t \in [0, T]$ , and are redundant.

We now consider two mild restrictions on the prices the firms can set over the horizon. First, it is assumed that firms price accordingly to ensure a positive market ( $M$ ). This relatively mild assumption is commonly imposed as a negative market share has no physical interpretation. Also, for the initial joint pricing and inventory problem, there are no inventory constraints as the inventory position at time  $t = 0$  will be determined after establishing the optimal equilibrium price path. As such, we assumed the price for firm  $i$  at any time  $t$  will be greater than or equal to the cost of inventory, i.e.

$P_i(t) \geq C_i$ . This ensures that the retailer does not explicitly plan on stocking and selling inventory for below cost. From (9) and the assumptions above, it must be that  $\lambda_{i,i}(t) \geq \eta_i \geq 0$  for all  $t \in [0, T]$ .

We can now establish the required sufficiency conditions for (8) to be an equilibrium point. In Jorgenson [34], it was shown that the above FOC's are sufficient if the following three conditions hold:

$$H_{X_i X_i}^i \leq 0 \quad H_{P_i P_i}^i \leq 0 \quad H_{X_i X_i}^i H_{P_i P_i}^i - (H_{X_i P_i}^i)^2 \leq 0$$

which establishes that the Hamiltonian for each firm is jointly concave in  $X_i(t)$  and  $P_i(t)$ , and are generally referred to as Mangasarian-type conditions. The third term in particular specifies that the Hessian of the Hamiltonian be negative semi-definite.

Proposition 1: For the differential game specified above, the Hamiltonians as given in (6) are jointly concave.

Proof. See Appendix A.  $\square$

Thus (8) is an open-loop Nash equilibrium solution for the proposed differential game. For ease of presentation, for the remainder of the analysis, we will restrict the market to a two-firm (duopoly) case. This allows one further simplification with respect to the notation. As  $\phi_i$  provides a relative measure of the response in both the market and market share equations, we can take firm i as the base firm, and normalize  $\phi_i = 1$  without any loss of generality in the model.

### 3.1. The Symmetric Firm Game

Though our primary interest involves firms with asymmetric market positions, we first analyze the symmetric game to gain insights from the sim-

pler symmetric firm case, in which all firms are equivalent in terms of market share, per unit inventory costs etc. Thus, if an equilibrium exists, the strategies will also be symmetric. As the initial state  $X_i(t)$   $i = 1, 2$  is equal for both firms, and equilibrium strategies are equivalent, we can see from (1) that  $\dot{X}_i(t) = 0$ , a constant or steady-state condition, resulting in  $X_i(t) = X_j(t) = 0.5$  for all  $t \in [0, T]$ . Thus, under symmetry, we can ignore the state variable in the equilibrium price path for both firms, which reduces to:

$$P_i^*(t) = \frac{2\bar{P}(t) + C_i}{3} - \frac{4\alpha\lambda_{i,i}(t)}{3\gamma} \quad (11)$$

Thus the pricing is in the form of a cost-plus (first term) pricing policy which is independent of the state (market share), and dependent only upon the adjoint variable  $\lambda_{i,i}(t)$ . Also, under symmetry we have a simplified form of the adjoint rate equation, specifically

$$\dot{\lambda}_{i,i}(t) = -(P_i(t) - C_i)\gamma(\bar{P}(t) - P_i(t)) \quad (12)$$

From (11) we can see that an upper bound restriction on the control set  $\mathcal{U}$ , required to maintain a positive market  $M(p)$ , is always met as  $P_i^*(t) \leq \bar{P}(t)$  for all  $t \in [0, T]$  in the symmetric game. Further, it is also obvious from (11), that  $P_i(t)^* \geq C_i$  as long as

$$\lambda_{i,i}(t) \leq (\bar{P}(t) - C_i)\frac{\gamma}{2\alpha}$$

resulting in an interior solution, while  $P_i(t)^* = C_i$  otherwise (i.e. boundary solution). Also, it is easy to show that if an interior solution exists at  $t = T$ , that is the residual value placed on market share ( $\eta$ ) is not excessive and meets the end requirement

$$\eta \leq (\bar{P}(T) - C_i)\frac{\gamma}{2\alpha} \quad (13)$$

then an interior solution is guaranteed to exist throughout the game. Details are given in Appendix B.

### *3.2. Solution Properties for the Symmetric Game*

We can now evaluate the solution of the symmetric game as well as any quantitative/qualitative properties. We first attempted to find closed form solutions for the adjoint equation (12), with the solution for  $\lambda_{i,i}(t)$  then be substituted into (11) to determine the equilibrium price path. Even for the simpler symmetric market, analytic results are only available where  $\bar{P}(t)$  is taken as a constant. However, for perishable value products, which are of primary interest, we expect  $\bar{P}(t)$  to be a strictly non-increasing, if not a strictly decreasing function of time, which precludes determination of a closed-form solution. Thus, even in the symmetric duopoly, we are required to rely upon numeric solutions. We therefore proceed with evaluating and interpreting the model parameters and their impact on the equilibrium price path.

We first examine the direction of the equilibrium price-path as a function of time. Taking the derivative of (11) with respect to time results in:

$$\dot{P}_i^*(t) = \frac{2\dot{\bar{P}}(t)}{3} - \frac{4\alpha\dot{\lambda}_{i,i}(t)}{3\gamma}$$

From this expression, it is obvious that price may be increasing or decreasing at any point of the sales horizon, dependent upon the expression specified for  $\bar{P}(t)$  and the resultant rate of change of  $\lambda_{i,i}(t)$  as given by (12). Thus no general conclusions can be drawn as to the direction of the price path, as it is model parameter specific.

We next evaluate  $\alpha$ , to determine the impact are of having a higher versus lower diffusion rate in the expression of market share. From (11), we observe that  $\alpha$  is only included in the third term of the equation. From the properties of the adjoint variables, specifically  $\lambda_{i,i}(t) \geq \eta \geq 0$ , this term is always negative as all parameters take on strictly positive values. Thus the equilibrium price  $P_i^*(t)$  is strictly decreasing in  $\alpha$  at any instant in time. Further, it is easy to demonstrate that equilibrium prices at higher  $\alpha$  values are always less than or equal to those obtained when lower  $\alpha$  values prevale in a given market, leading to the following proposition:

Proposition 2: Equilibrium prices decrease along the price path with increasing  $\alpha$ .

Proof. See Appendix C.  $\square$

From this, we can observe that increasing  $\alpha$  results in a more competitive market, i.e. a more responsive market where consumers are more aware of and/or more quickly react to price differentials, which forces both firms to reduce prices along the equilibrium path. Specifically, as  $\alpha$  approaches infinity, prices are reduced to the lower boundary of the control space  $\mathcal{U}$ , i.e. to cost for both players, or more likely to cost plus some acceptable margin. It is obvious that the margin would be equivalent for both firms, as any firm deviating by trying to unilaterally increase its per unit contribution would lose its market share. It is therefore possible to interpret  $\alpha \rightarrow \infty$  as a perfectly competitive market.

Finally, we evaluate the effects of  $\gamma$  on the equilibrium price  $P^*(t)$ . As  $\gamma$  effectively scales the size of the available market, all else being equal, a larger  $\gamma$  results in a larger potential market, where it is anticipated that

prices would be higher than in a smaller (more competitive) market. However, it turns out that this only applies when  $\eta$  is greater than zero, with no impact on the equilibrium price path at  $\eta = 0$ . This is stated in the following proposition:

Proposition 3:  $\gamma$  has no effect on optimum prices when the residual or salvage value of market share ( $\eta$ ) is 0, while prices are weakly increasing for increasing  $\gamma$  when  $\eta > 0$ .

Proof. Details of the proof for this proposition are contained in Appendix D.  $\square$

### 3.3. The Asymmetric Game

We now turn our attention to the more interesting scenario and analyze the game when the players are asymmetric. This would be typical of the situation where a smaller retailer, which we will denote as firm  $j$ , such as a local independent or regional chain competes against a dominant or big-box retailer (firm  $i$ ). Questions of interest would include: 1) under what conditions does an equilibrium exist; 2) what practical strategies exist given differences in inventory costs, price perceptions (as determined by  $\phi_j$ ), inventory restrictions (due to capital or space constraints), as well as possible limits on market share (for example, the maximum encroachment firm 'i' will tolerate on its market position). In general, we will assume that firm 'j' has an equal or higher cost per unit inventory for an identical product line to firm 'i'. As well, we assume by definition that the initial market share of firm 'j' ( $X_{j,0}$ ) is less than that of the larger retailer.

Within the control constraint  $\mathcal{U}$  defined previously, the conditions under

which an equilibrium doesn't exist can generally be understood as those that prevent firm 'j' from maintaining a positive market share ( $X_j(t) \geq 0$ ) for a given price path  $P_j(t) = C_j, t \in [0, T]$ . It is difficult to give specific conditions under which this occurs as various parameter combinations could enable this result. However, it is possible to express conditions that provide for a solution to exist. The most general setting or condition would allow us to determine where, as firm j priced at its minimum level ( $C_j$ ) with  $X_j \rightarrow 0$ , we maintained  $\dot{X}_j(t) > 0$  and  $X_j(t) > 0$ . This would, in general, be a function of the equilibrium price for firm i ( $P_i^*(t)$ ), as well as the parameters  $\alpha$  and  $\phi_j$  and the length of the sales horizon. However, we would prefer to express the requirements purely in terms of the model parameters and initial conditions. To accomplish this, we impose the following requirement which ensures that equilibrium conditions exist and all state requirements are met:

$$X_{j,0} - \alpha(C_i - \phi_j C_j)T \geq 0 \quad (14)$$

where  $\alpha(C_i - \phi_j C_j)T$  is the maximum change in market share for firm j over the sales horizon (T). Thus the above inequality simply states that as long as this decrease in market share for firm j over the sales horizon is less than the initial market position, then the non-negativity requirement will be assured.

We proceed with our analysis of the solution for the asymmetric game under the assumption that a solution exists based upon the above requirements, or more specifically that parameter values exist that allow for an equilibrium solution. We first discuss the existence of a Nash equilibrium without market or other restrictions. For the asymmetric game, the equation for the equilibrium price paths for two firm's 'i' and 'j' are as given in (8).

### 3.4. Solution Properties for the Asymmetric Game

As with the symmetric game we attempt to discern qualitative properties of the model, as no analytic solution exists for the equilibrium price paths.

As above, there are no determination of the direction of the equilibrium price-paths given the competing time-dependent variables that impact  $P_i^*(t)$ , particularly in the asymmetric game. From the time derivative of the equilibrium price paths  $\dot{P}_i^*(t) = \dot{P}(t) - \frac{\alpha(\dot{\lambda}_{i,i}(t))}{\gamma X_i(t)} + \frac{\alpha \lambda_{i,i}(t) \dot{X}_i(t)}{\gamma X_i(t)^2} - \frac{\phi_j}{2} \dot{P}_j(t)$  it is obvious that no general determination of the direction of the prices for the firms can be made over the sales horizon. This is to be expected as it was similarly not possible to do so even in the conceptually simpler symmetric game.

From (1) it is obvious that  $\alpha$  is critical in determining whether an equilibrium exists. Evaluating the game at the limits, we see that an equilibrium always exist as  $\alpha \rightarrow 0$ , with prices going to steady-state levels:

$$P_i(t) = (2\bar{P}(t) + 2C_i - C_j)/3 \quad i = 1, 2 \quad i \neq j$$

with the mark-up or profit per unit inventory for firm 'j' always less than that for the larger firm 'i'. Conversely, as  $\alpha \rightarrow \infty$ , a solution can only exist under the unique condition of  $\phi_i P_i(t) = \phi_j P_j(t)$ , or equivalently  $(2\phi_i + \phi_j)C_i = (2\phi_j + \phi_i)C_j$ . This indicates that a specific relationship must exist between the cost and price advantage, with the added cost per unit inventory of firm 'j' balanced by a pricing advantage, with firm 'j' able to charge a higher price while maintaining a positive market share. That is, if no market advantage exists for firm 'j', as defined by  $\phi_j < \phi_i$ , then an equilibrium cannot exist as the firm cannot maintain a positive market share. Thus under increased competition (higher  $\alpha$ ), it becomes increasingly difficult for

the smaller firm to compete in a market place dominated by a big-box retailer without differentiating along product lines (different cost structure) and/or service (lower  $\phi_j$ ).

This is often observed in numerous markets, with two common approaches available for the smaller firm. One is to compete with high-end products and service, thus providing for a profit niche with products that command a premium price. Exclusive retailers of high-end consumer electronics would be an example of this approach. The other method is to offer products with a cost advantage that permits a profitable niche at the low-end of the market. A final approach is to distinguish yourself by location, service, etc. in a way that you can maintain a small albeit positive market share given a higher cost structure and price premium. This would be equivalent to reducing  $\phi_j$  in our model. Even when a solution exists and firm 'j' is able to make a profit from sales of identical products as a much larger retailer, it is quite evident that there may exist product lines that would offer player 'j' increased profit margins using one of the approaches presented.

### *3.5. Numeric Algorithm*

Given the limitations in solving the asymmetric game analytically, to evaluate the outcomes and gain insights into the structure of the pricing and inventory policies it is necessary to develop procedures to numerically solve the problem. The numeric solution of the open-loop Nash equilibrium for the differential game can be approached similar to that for deterministic optimal control problems. Specifically, we need to numerically approximate a system of ordinary differential equations, consisting of a two-point boundary value problem. That is, at time  $t = T$ , we have the following known conditions:

$\lambda_{i,i} = \eta_i$  and  $\lambda_{j,j} = \eta_j$ , while at time  $t = 0$ , we assume we know  $X_{i,0}$  and  $X_{j,0}$ . Using a solution algorithm that starts at  $t = T$ , we make an initial guess of  $X_i(T)$  and  $X_j(T)$ . A reasonable initial approximation may be to use the initial market share values. The primary difference between the solution of the differential game and an equivalent optimal control problem involving a single decision maker is the simultaneous determination of multiple decision controls (prices). In particular, the solutions as calculated by (8) need to be within the defined control set  $\mathcal{U}$ . If an interior solution exists, the algorithm proceeds sequentially backwards in time in the same manner as in an optimal control approximation. However, if one (or both) of the prices fall outside  $\mathcal{U}$ , then a boundary solution at that instant in time may exist, and the equilibrium point is recalculated for both players until the equilibrium point is found. A detailed description of the numeric algorithm is contained in Appendix E.

## 4. Results and Discussion

The following section presents some numeric results and in particular highlights the impact of the various model parameters on the price and inventory policies for both firms, as well as realized profits over the sales horizon. As well, we emphasize and discuss the implications of moving from the assumption of symmetric to asymmetric firms in the market and the impact of policies restrictions on the policies and profits for the smaller firm.

### 4.1. Symmetric Firm Results

We will begin by evaluating the results for two symmetric firms. As the firms in a symmetric duopoly by definition have the same cost (per unit

inventory) and product valuations ( $\phi$  values), the primary factors to consider are of the parameters  $\alpha$  and  $\gamma$ . The results are scaled to results generated at the minimum  $\alpha$ , with results for this solution normalized to 1.

#### *4.1.1. Effect of $\alpha$ in a symmetric duopoly*

Results over a range of  $\alpha$  values are presented in figures 1 and 2 showing the effect of  $\alpha$  on pricing as well as sales/inventory levels and profits. As the firms are symmetric, the results are equivalent for both. From the figures, we observe a few trends. First, prices, both initial and average, are decreasing in  $\alpha$ . Average prices as well as the average return per unit inventory is shown graphically in Figure 1. We see that the return on inventory, given on the left hand axis approaches zero, and the average price goes to inventory cost (right-hand axis) as  $\alpha$  increases. As would be expected, sales increase as  $\alpha$  increases as a result of the lower prices in the market; however, the increasing sales are insufficient to compensate for the reduced return per unit inventory. This is consistent with the interpretation of  $\alpha$  as a measure of market competitiveness, through consumer response to price differentials. The mechanism can be taken as both awareness of price differentials as well as how rapid the response to this information in the market.

Sales as a function of  $\alpha$  are given in Figure 2. From Figure 2 we see that sales are strictly increasing as  $\alpha$  increases. This is direct response to the downward pressure on prices noted above. As market prices decline, the overall market size increases, resulting in increased sales, albeit at a lower price.

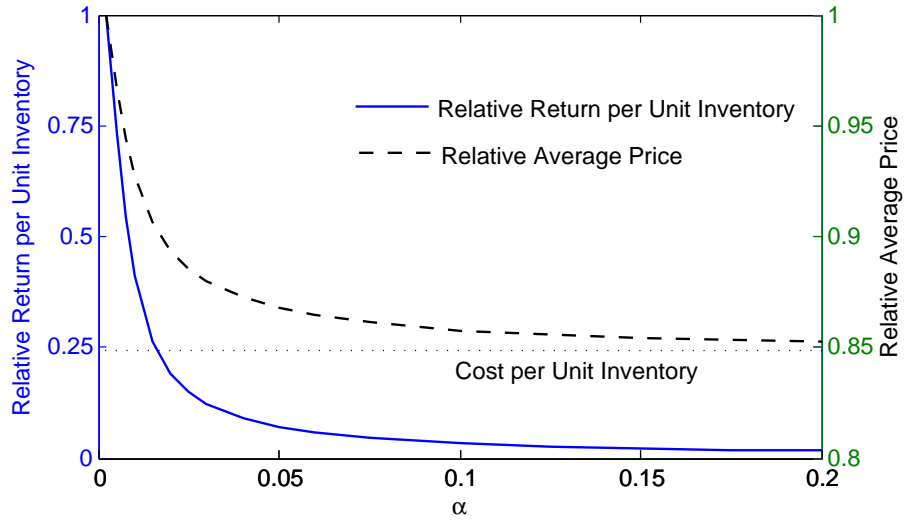


Figure 1: Impact of  $\alpha$  on Average Prices & Returns

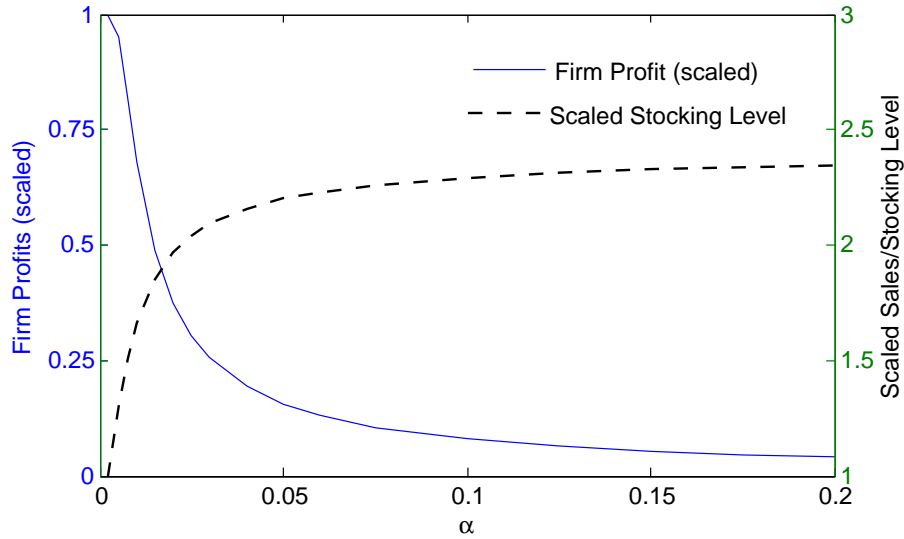


Figure 2: Firm Profits & Sales as a function of  $\alpha$

#### 4.1.2. Market size ( $\gamma$ ) and residual values ( $\eta$ )

From Proposition 3, we know that  $\gamma$  has no effect on prices when no residual value ( $\eta$ ) is assigned to market share, while sales linearly increase

with  $\gamma$ . However, once residual values are introduced, we see the effects of  $\gamma$  on prices as well as sales. Therefore, the results for  $\gamma$  are presented in conjunction with the residual market value. Table 2 shows the price, sales and average returns per unit inventory over a range of  $\eta$  and  $\gamma$  values for  $\alpha = 0.004$ . The results are scaled relative to results at  $\eta = 0$  and  $\gamma = 1$ . The value for  $\eta$  is also scaled, with 1 or the maximum being the value of  $\eta$  that results in boundary solutions, i.e prices equal to inventory cost, for the maximum  $\gamma = 1.5$ . This is based upon the conditions as specified in equation (13). An  $\eta$  of zero simply indicates that no residual value was considered. Figure 3 below graphically illustrates the interaction between  $\gamma$  and the salvage value  $\eta$  for a fixed  $\alpha$  and the impacts on average firm prices over the sales horizon.

Table 2: Effect of  $\gamma$  and  $\eta$  in symmetric duopoly

$\eta$	$\gamma$	Price			Sales			Avg Return		
		0.50	1.0	1.5	0.50	1.0	1.5	0.50	1.0	1.5
0	0.5	1	1.5	0.5	1	1.5	1	1	1	
0.25	0.102	0.748	1.359	1.154	1.886	2.489	0.088	0.397	0.546	
0.50	0	0.140	0.765	1.204	2.343	3.163	0	0.060	0.242	
0.75	0	0	0.241	1.204	2.407	3.498	0	0	0.069	
1	0	0	0	1.204	2.407	3.611	0	0	0	

As would be expected, we see average prices decrease as greater emphasis is placed on future sales/market share, as indicated by higher residual values. This effect is stronger when the initial market size or scale ( $\gamma$ ) is smaller, which increases the relative importance of maintaining market share

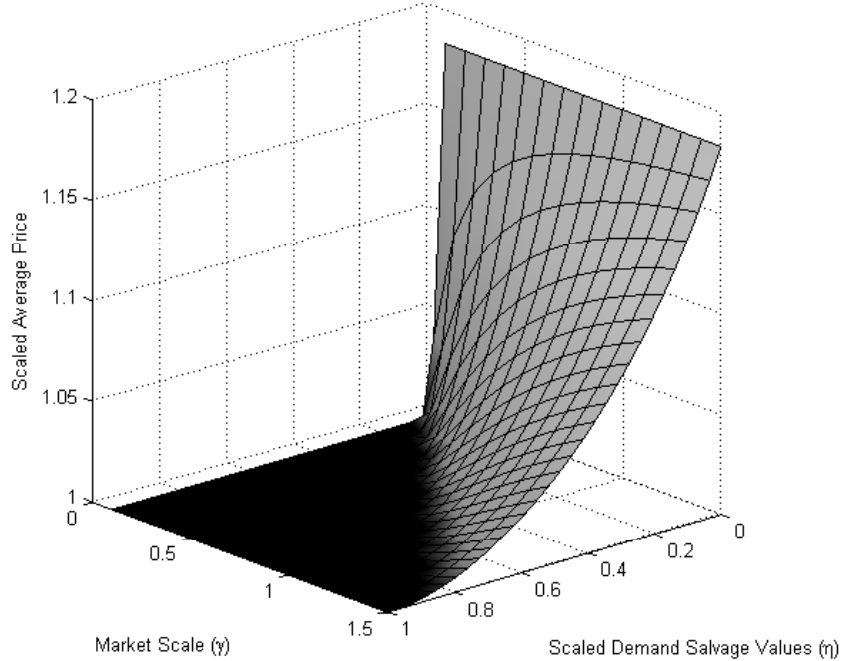


Figure 3: Impact of  $\gamma$ ,  $\eta$  on Average Prices

for future sales and resultant profits. As the scale of the available market increases, the emphasis is placed on more immediate sales and profit taking over the sales horizon.

#### 4.2. *Asymmetric Firm Results*

We now evaluate the model results for a duopoly situation where we have asymmetric firms competing in a common market. We are particularly interested in the model behavior and strategies for the smaller firm competing with much larger rivals, and the interactions of market parameters on the preferred strategy. The asymmetric firms start with different market shares.

For our analysis, we selected  $X_{1,0} = 80\%$  and  $X_{2,0} = 20\%$ , thus there is initially a four-fold difference in the markets for each firm. Limits on  $P_2$  are established by specifying a maximum change in the initial market share of Firm 2 on an annual basis. For most results, this was taken as 20% per year, or double the firm's initial market share.

To facilitate both the presentation and discussion, the cost for Firm 2 is listed in combination with the  $\phi_2$  value relative to the cost of inventory for Firm 1. Thus, in equilibrium  $C_2/\phi_2 = C_1$ , which is given in the tables and figures as  $\beta = C_2/(\phi_2 C_1) = 1$ , and neither firm enjoys a product valuation advantage. Results less than or greater than 1 indicate a valuation advantage or disadvantage respectively for Firm 2. The total  $\beta$  range tested was 0.96 to 1.04.

#### *4.2.1. Market and Firm Parameters: $\alpha$ and $\beta$*

The diffusion or 'market' parameter  $\alpha$  defines how rapidly market share responds to inequalities in the price relative to the value. This difference in valuation is expressed through the  $\phi$  for each firm and its products relative to the per unit inventory cost, and is expressed as  $\beta$  as described above.

The  $\alpha$  term is a key parameter in the model, both from an intuitive interpretation and understanding of its impact on the market share of each firm and the type of market in which they exist. As described in the previous section, strategically,  $\alpha$  can be interpreted in one of two ways: as a measure of rate of consumer information or knowledge of the price differentials or as a general measure of the overall competitiveness of the market itself. The  $\phi$  variable, and by extension  $\beta$ , is used to allow comparison of differentiated product offerings.

The results are compared across a range of values, as well as at different product offerings from Firm 2 as determined by inventory costs. Specifically Firm 2 is taken as having low, equivalent, and high end merchandise relative to the larger Firm 1, with related inventory costs for the type of product. For the discussion below, the results in the following tables and figures are taken at equivalent inventories costs(i.e.  $C_1 = C_2$ ). Comparable results are obtained for low and high Firm 2 inventory costs with similar  $\alpha$  and  $\beta$  values. The results are given relative to a base case taken at the minimum  $\alpha$ . Resultant sales and average prices are presented in Table 3.

From Table 3 we observe that average prices fall as  $\alpha$  increases, similar to the results obtained for the symmetric duopoly. Also we can see where no solution to the game existed, as Firm 2 was not able to maintain a positive market share. In the tables, this is identified by the lack of value for a given set of parameters. As would be expected, this is more prevalent in combinations of increased competition (higher  $\alpha$ ) and a Firm 2 price disadvantage (higher  $\beta$ ). The impact on sales/inventory levels however differ for the two firms depending upon the combination of  $\alpha$  and  $\beta$ . Results for Firm 1 show increasing sales/optimum inventory levels as  $\alpha$  increases as a consequence of lower overall prices. This trend is consistent across all values of  $\beta$ . The same results are observed for Firm 2 under favorable  $\beta$  conditions, that being  $\beta < 1$ . However, at higher  $\alpha$ 's and  $\beta > 1$ , sales can begin to decrease as Firm 2 struggles to maintain a positive market share. Thus the market advantages enjoyed by Firm 1 in market share and the pricing disadvantage (higher  $\beta$ ) faced by Firm 2 in a more competitive market (higher  $\alpha$ ) allow the larger firm to push the smaller firm out of the market.

Table 3: Effect of  $\alpha$  and  $\beta [= C_2/(\phi_2 C_1)]$  on Pricing & Sales

$\alpha$	$\beta$	Avg P					Sales				
		0.968	0.984	1	1.016	1.032	0.968	0.984	1	1.016	1.032
<u>Firm 1</u>											
0.001		1	1	1	1	1	1	1	1	1	1
0.005		0.853	0.852	0.850	0.847	0.846	1.573	1.583	1.589	1.596	1.602
0.01		0.778	0.775	0.774	0.771	0.770	1.901	1.907	1.909	1.912	1.915
0.02		0.729	0.727	0.725	0.723	0.722	2.111	2.113	2.109	2.111	2.141
0.03		0.711	0.709	0.708	0.705	0.711	2.190	2.188	2.184	2.184	2.670
0.05		0.696	0.694	0.692	0.692	-	2.256	2.254	2.248	2.592	-
0.075		0.688	0.686	0.684	-	-	2.291	2.288	2.281	-	-
0.125		0.680	0.678	0.677	-	-	2.320	2.316	2.309	-	-
<u>Firm 2</u>											
0.001		1	1	1	1	1	1	1	1	1	1
0.005		0.887	0.882	0.876	0.870	0.865	1.542	1.551	1.563	1.578	1.592
0.01		0.813	0.807	0.802	0.797	0.791	1.873	1.877	1.890	1.899	1.911
0.02		0.765	0.760	0.754	0.749	0.744	2.100	2.100	2.112	2.116	2.013
0.03		0.747	0.741	0.736	0.731	0.736	2.189	2.187	2.197	2.203	0.433
0.05		0.731	0.727	0.721	0.720	-	2.266	2.263	2.271	1.068	-
0.075		0.723	0.718	0.712	-	-	2.308	2.304	2.310	-	-
0.125		0.715	0.711	0.706	-	-	2.343	2.339	2.345	-	-

#### 4.2.2. Inventory Strategies: Firm 2

A particularly interesting question is what strategy with respect to inventory a smaller retailer should adopt when faced with competition from a much larger big-box retail firm. To evaluate this, we take the smaller of the two firms at three different product types, based upon associated costs of inventory. These are equivalent to the larger retailer ( $C_{2,e}$ ), as well as high end ( $C_{2,h}$ ) and low end ( $C_{2,l}$ ) merchandise. We then compare the realized profits for a specific example across a range of relative product valuations

( $\beta$ ) and market competitiveness ( $\alpha$ ), selecting the situation with the highest profits. The results of this selection process is given in Table 4. As in the previous tables, no results indicate where a solution to the game does not exist.

Table 4: Inventory strategy as a function of  $\alpha$  and  $\beta$

	$\beta$										
$\alpha$	0.96	0.968	0.976	0.984	0.992	1	1.008	1.016	1.024	1.032	1.04
0.0001	Low	Low	Low	Low	Low	Low	Low	Low	Low	Low	Low
0.001	Low	Low	Low	Low	Low	Low	Low	Low	Low	Low	Low
0.005	Eq	Eq	Eq	Eq	Eq	Eq	Eq	Eq	High	High	High
0.01	High	High	High	High	High	High	High	High	High	High	High
0.02	High	High	High	High	High	High	High	High	High	High	-
0.03	High	High	High	High	High	High	High	High	-	-	-
0.05	High	High	High	High	High	High	High	High	-	-	-
0.075	High	High	High	High	High	High	High	-	-	-	-
0.125	High	High	High	High	High	High	-	-	-	-	-
0.175	High	High	High	High	High	High	-	-	-	-	-

Low = low end stock; Eq = equivalent to Firm 1; High = premium priced inventory

From Table 4, we first notice how infrequent an equivalent inventory is the preferred selection, with differentiation along product lines usually resulting in increased profits for Firm 2. Generally, under reduced competition a lower valued/priced product is preferred while high end inventories generate higher profits when competition increases. Thus, for smaller firms, an understanding of the competitive environment, particularly when facing competition from much larger rivals can help direct the strategy with respect to the type of inventory and the position the firm takes.

### 4.3. Policy Restrictions: Limits on $P_2$

As noted during the specification of the differential game, we have used restrictions on the price Firm 2 can charge to establish 'logical' bounds on the market and the interactions of the two firms. To assess the importance of these bounds, we compare a set of results in which the pricing limits on Firm 2 are removed. We look at the sales and profit implications and discuss how the bounds are required to maintain the structure of the game and provide realistic results when dealing with asymmetric firms and markets. The results presented are for equivalent inventories ( $C_1 = C_2$ ) and with  $\beta = 1$ , so no price or inventory advantage is included.

Taking three  $\alpha$  values, the results for both firms are presented in Table 5 below, showing final profits, sales, as well as market share at the end of the sales horizon. For Firm 2, the unconstrained profits are as expected always higher than when its pricing is limited by  $P_{2,l}$ . For lower values of  $\alpha$ , the impact of  $P_{2,l}$  is relatively small. For these market conditions, no explicit bounds on the firm's pricing strategy are required. However, as  $\alpha$  increases, we see that Firm 2 quickly begins to close the gap on Firm 1 in both sales and market share even though its initial market share is taken as a quarter that of the larger firm. Firm 2 eventually increases its final market share to equivalent to that of Firm 1. Firm 1's profit also begins to decline, and it is unrealistic to assume that Firm 1 would not use its much larger market position to protect its initial market share. Thus the use of pricing bounds, particularly in markets that are more competitive, are effective in establishing practical limits on the pricing strategies when the firms are asymmetric, and provide for more realistic outcomes.

Table 5: Effect of  $P_2$  Limits

	$\alpha$	Mode	Profit	Sales	Market Share
Firm 2	0.0005	Unconstrained	12,123	11	28.9 %
		Constrained	11,773	9	25.8 %
	0.001	Unconstrained	15,007	16	39.9 %
		Constrained	12,647	10	27.7 %
	0.005	Unconstrained	12,653	34	50.0 %
		Constrained	10,059	16	29.7 %
Firm 1	0.0005	Unconstrained	52,335	30	71.1 %
		Constrained	49,648	30	74.2 %
	0.001	Unconstrained	51,074	31	60.1 %
		Constrained	48,796	33	73.3 %
	0.005	Unconstrained	24,209	42	50.0 %
		Constrained	34,957	52	70.3 %

#### 4.4. Pricing Strategy: Firm 1

As discussed in the general results above, under highly competitive situations, as represented by high  $\alpha$  values, profits for both firms tend to fall as a result of downward pressures on prices. Again, the assumption is that price collusion between the two firms is not allowed or does not occur. However, as noted in general strategy texts on firm pricing, for example those by Marn et al. [35] and Nagle and Holden [36], there are acceptable techniques available for firms to take a ‘leadership’ role in setting prices and improve overall profits. This, in particular, would be advantages as  $\alpha$  increases. Under proper conditions, it would be more profitable say for Firm 1 (the larger firm) to take a leadership role in setting its price higher than that which would be determined under equilibrium. The risk of course is that Firm 2 could, within

a highly competitive market, take advantage and initially gain a much larger market share at the expense of the rival firm. However, as Firm 1 would enjoy both a favorable market share and likely a cost advantage as well, Firm 2 could be quickly ‘punished’ in any ensuing price war.

A simple example to illustrate the results of assuming Firm 1 takes the lead and establishes the general pricing strategy for the market, which is then picked up by Firm 2 is given below. For simplicity, we use equivalent inventory/costs for both firms, though similar results are generated for differentiated products. We will begin by stating that Firm 1’s expectation is that Firm 2 will accept and maintain its current market share, or conversely that Firm 2 understands that Firm 1 will not allow any loss of market share. Thus, we can set  $P_2(t) = P_1(t)/\phi_2$ . Solving for the optimum  $P_1(t)$  under these conditions results in

$$P_1^*(t) = \frac{\phi_2(2\bar{P}(t) + C_1)}{2(\phi_2 + 1)} + \frac{C_1}{2(\phi_2 + 1)}$$

These prices can then be substituted into the value function and sales equations for both firms and integrated to determine the resultant inventory levels and profits for these policies. Using  $\alpha = 0.005$ , we can compare the results to those presented in Table 5 above. For the given parameters, we find that profits for Firms 1 and 2 rise by 38.8 % and 20.6 % while inventory levels decrease by 21.1 % and 37.5 % respectively. Thus, Firms 1 and 2 increase profits on lower sales under this arrangement, making for an effective strategy for both firms.

## 5. Conclusions

The model presented in this paper uses differential equations to model the evolution of the demand capture for two firms in duopoly, in which the market capture rate, and therefore the resultant demand rate for each firm, is path-dependent. The firms can be symmetric or asymmetric. Within the asymmetric duopoly, we establish practical bounds on the solutions to provide for realistic results. Solution of the equilibrium prices for both firms were computed, with resultant inventory levels and firm profits determined. The paper evaluates the impact and the managerial implications of the model parameters on the policies of both firms. Specific strategies are developed for both firms resulting from different market conditions as determined by the model parameters under both symmetric and asymmetric competition. For the smaller firms facing competition from big-box retailers, we provide insight and direction into the strategies with respect to product lines based upon the competitive landscape. For the larger firms, we evaluate the role of leadership in pricing, particularly in more competitive markets.

There are several directions currently being pursued in conjunction with this research. The first is the extension of the model to incorporate uncertainty in demand, either with respect to the rate of demand capture or the resulting sales. There is also the role of additional and impact of additional marketing mix controls, particularly advertising. The results are also being extended to include more than two firms in the market. A final extension would be establishing different restrictions on the policies of the smaller firm in asymmetric duopolies, such as maximum sales/inventory.

## Appendix A: Proof of Proposition 1

### Proof that $H_i$ is a concave function

Given the Hamiltonian in equation (6)

$$H_i = (P_i - C_i)M(t)X_i + \lambda_{i,i} \alpha \left( \sum_{j \neq i} \frac{\phi_j P_j}{N-1} - \phi_i P_i \right) + \sum_{j \neq i} \lambda_{i,j} \alpha \left( \sum_{k \neq j} \frac{\phi_k P_k}{N-1} - \phi_j P_j \right)$$

The Hamiltonian,  $H_i$ , is a concave function in its arguments  $P_i$  and  $X_i$  if:

$$H_{X_i X_i}^i \leq 0$$

$$H_{P_i P_i}^i \leq 0$$

$$H_{X_i X_i}^i H_{P_i P_i}^i - (H_{X_i P_i}^i)^2 \leq 0$$

Taking the required partial derivatives results in:

$$H_{X_i X_i}^i = 0$$

$$H_{P_i P_i}^i = -\gamma \phi_i X_i < 0$$

$$H_{X_i X_i}^i H_{P_i P_i}^i - (H_{X_i P_i}^i)^2 = 0 - \left[ \gamma (\bar{P}(t) - \frac{\phi_i P_i + \phi_j P_j}{2}) - \frac{(P_i - C_i)\gamma}{2} \right]^2 \leq 0$$

Thus it is obvious that the all three required conditions for concavity of the Hamiltonian ( $H_i$ ) are met. Equivalent results are obtained for  $H_j$ .

## Appendix B

### Proof of an interior solution for a symmetric game given conditions on $\eta$

Taking a known point,  $t = T$ , we have:

$$P_i^*(T) = \frac{2\bar{P}(T)}{3} + \frac{C_i}{3} - \frac{4\alpha\eta}{3\gamma}$$

Thus, an interior solution at  $t = T$  will exist, i.e.  $P_i \geq C_i$  as long as

$$\eta \leq (\bar{P}(T) - C_i) \frac{\gamma}{2\alpha}$$

For a perishable value asset, if  $\eta$  meets this requirement at  $t = T$ , this inequality is valid for all  $t$  as we take  $\dot{P}(t) \leq 0$ , i.e. a strictly non-increasing function of time.

If  $P_i^*(T) = C_i$ , then going from  $T$  to  $T - \delta$  then the instantaneous rate of change for  $\lambda_{i,i}$  is 0 given (9), thus  $P_i^*(T - \delta) \geq C_i$  as  $\delta \rightarrow 0$ . Specifically,  $P_i^*(T - \delta) = C_i$  for  $\dot{P}(t) = 0$ , and greater than  $C_i$  otherwise. This condition will remain valid as  $t$  goes backward to 0. Similarly, if at  $P_i^*(T) > C_i$  but equals  $C_i$  somewhere on the reverse path  $t \rightarrow 0$ , then  $P_i^*(t)$  will again remain on or above the boundary  $C_i$  for all  $t \in [0, T]$ .

## Appendix C: Proof of Proposition 2

### Proof that equilibrium prices decrease with increasing $\alpha$

To determine if this holds over the entire time horizon we begin at time  $t = T$ . Obviously  $P_i^*(T)$  will be lower for higher values of  $\alpha$  for any positive salvage  $\eta$ , and equivalent if the residual value is taken as 0.

We begin the analysis under the condition of  $\eta = 0$ , and expand the results below. Take two values,  $\alpha$  and  $\alpha'$ , with  $\alpha' > \alpha$ . Further, define the resulting equilibrium paths as  $P_i^{*'}(T)$  and  $P_i^*(T)$  respectively.

At  $t = T$ , the following holds true:

- (1) from (11)  $P_i^{*'}(T) = P_i^*(T)$ .
- (2) from transversality conditions  $\lambda'_{i,i}(T) = \lambda_{i,i}(T) = \eta = 0$
- (3) and from (12)  $\dot{\lambda}'_{i,i}(T) = \dot{\lambda}_{i,i}(T)$

Taking  $\delta \rightarrow 0$ , for  $t = T - \delta$ , with the rate of change in  $P_i^*$  and  $P_i^{*'}$  given by

$$\dot{P}_i^* = \frac{2\dot{\bar{P}}(t)}{3} - \frac{4\alpha\dot{\lambda}_{i,i}}{3\gamma}$$

it is obvious that  $P_i^{*'}(t) < P_i^*(t)$  as  $\alpha'\dot{\lambda}'_{i,i}(T) > \alpha\dot{\lambda}_{i,i}(T)$

However, this condition does not generally hold for all  $t$  in  $[0, T]$  as equation (12) is concave in  $P_i$ , with the rate of change being exactly zero at either extreme of the control space  $\mathcal{U}$ , i.e.  $C_i$  and  $\bar{P}(t)$ , with the maximum instantaneous rate occurring at  $(C_i + \bar{P}(t))/2$ . It is therefore feasible for the direction of  $P_i^{*'}$  to change and approach  $P_i^*$  at some  $0 < t < T$ . Define this first approach as occurring at  $t = \tau$ . However as  $P_i^{*'}(\tau) \rightarrow P_i^*(\tau)$ , by definition  $\dot{\lambda}'_{i,i}(\tau) \rightarrow \dot{\lambda}_{i,i}(\tau)$ , resulting in  $\alpha'\dot{\lambda}'_{i,i}(\tau) > \alpha\dot{\lambda}_{i,i}(\tau)$ , thus causing the prices to diverge again. This process can be repeated over the remaining reverse path  $t \in [\tau, 0]$ , ensuring that  $P_i^{*'}(t) \leq P_i^*(t) \quad \forall t \in [0, T]$ .

Equivalent arguments occur for  $\eta > 0$ , with the only (simplifying) condition being that the final price for  $\alpha'$  at  $t = T$  is already less than  $P_i^*(T)$ , that is from (11)  $P_i^{*'}(T) < P_i^*(T)$  as  $\lambda'_{i,i}(T) = \lambda_{i,i}(T) = \eta > 0$  and  $\alpha'\eta > \alpha\eta$ .

## Appendix D: Proof of Proposition 3

**Proof that equilibrium prices increase with increasing  $\gamma$  for  $\eta > 0$ , and independent of  $\gamma$  at  $\eta = 0$**

Define  $\gamma' > \gamma$ .

For  $\eta = 0$ , we have  $P_i^{*'}(T) = P_i^*(T) = \frac{2\bar{P}(T)}{3} + \frac{C_i}{3}$ . From (12) it is obvious for equal prices that  $\dot{\lambda}'_{i,i}(T) > \dot{\lambda}_{i,i}(T)$ , but also that  $\dot{\lambda}'_{i,i}(T)/\gamma' = \dot{\lambda}_{i,i}(T)/\gamma$ .

Equivalently, with the rate of change in  $P_i$  given by

$$\dot{P}_i^* = \frac{2\dot{P}(t)}{3} - \frac{4\alpha\dot{\lambda}_{i,i}}{3\gamma}$$

we have  $\dot{P}'_i(T) = \dot{P}_i(T)$ . Taking  $\delta \rightarrow 0$ , for  $t = T - \delta$ , it is obvious that  $P_i^{*'}(t) = P_i^*(t)$ , resulting in  $\dot{\lambda}'_{i,i}(t)/\gamma' = \dot{\lambda}_{i,i}(t)/\gamma$  and the equivalent  $\dot{P}'_i(t) = \dot{P}_i(t)$ , which will obviously hold for  $t \in [0, T]$ , and result in  $P_i^{*'}(t) = P_i^*(t)$  for all  $t$ .

For  $\eta > 0$ , we have end conditions  $P_i^{*'}(T) > P_i^*(T)$  given equation (11) and  $\lambda'_{i,i}(T) = \lambda_{i,i}(T) = \eta$ .

Similar to the previous proposition, it is possible that the two prices will begin to converge at some time  $t = \tau \in [0, T^-]$ . As  $P_i^{*'}(\tau) \rightarrow P_i^*(\tau)$ ,  $\frac{\dot{\lambda}'_{i,i}(\tau)}{\gamma'} \rightarrow \frac{\dot{\lambda}_{i,i}(\tau)}{\gamma}$ , and  $P_i^{*'}(t) = P_i^*(t)$  for  $t \in [0, \tau]$ . Therefore  $P_i^{*'}(t) \geq P_i^*(t) \forall t \in [0, T]$ .

## Appendix E: Numeric Algorithm

The method is basically a shooting type algorithm, with one modification. Instead of initializing a single 'estimate' of  $X_1(T)$ , with the resultant  $X_2(T)$ , a pair of values are initialized and calculated simultaneously. The two values are generated from an initial guess (say 0.5) denoted as  $X_e$ , and then taken as  $\pm\delta$ , where  $\delta$  is user defined (say 0.1). This approach was found to run faster and converge more quickly than a single forward-type shooting algorithm.

**Initialization** Given  $\lambda_{i,i}(T) = \eta_i$ ,  $i = 1, 2$ , initialize  $t = T$  and set time step  $= \delta t$  and Parameter Values  $C_i$ ,  $\phi_i$ ,  $\alpha$ ,  $\bar{P}$ ,  $\gamma$ ,  $\eta_i$ ,  $P_l$ ,  $\delta$ , and  $X_{i,given}$  (initial market share).

Set maximum number of iterations.

**Step 1**

Provide initial estimates of final value of the state variables  $X_e$ .

Creates pair of estimates (from  $\pm\delta$ ) denoted as  $X_i^A(T)$  and  $X_i^B(T)$

Set accuracy for stopping criteria:  $|X_i(0) - \eta_i| \leq \epsilon$

**For each initial estimate, perform the following steps as indicated:**

**Step 2**

Start at  $\tau = 0$  ( $t=T$ )

Use equation (8) to calculate the equilibrium prices  $P_i^*(T)$ ,  $i = 1, 2$

**Step 3**

Determine if both firms prices are within control set ( $\mathcal{U}$ ), go to Step 5

**Step 4**

Perform iterative procedure for  $P_i$ ,  $i=1,2$  to find equilibrium prices that satisfy  $\mathcal{U}_i$ .

**Iteration Procedure:**

The price outside the control set is set to the lower bound, if the price is outside  $\mathcal{U}_i$ . The equilibrium price for the other firm is then recalculated. If no internal solution for either firm exists, then both prices are set to boundary values:  $P_1 = C_1$ ,  $P_2 = \max[C_2, P_{2,limit}]$ .

**Step 5**

Increment time and update state and adjoint variables:  $t = t - \delta t$ ,  $X_i(t)$ ,  $\lambda_{i,i}(t)$

**Step 6**

Check validity conditions on  $X_i$  and end time  $t=0$ :

If  $X_i(t) < 0$  then

Record number of steps ( $t = t_s^A$  for loop A and  $t = t_s^B$  for loop B) and end loop

If  $X_i(t) > 0$  and  $t=0$ , record  $X_i(0)$  and proceed to Step 8

Otherwise, return to Step 2 and repeat calculations.

**Step 7**

Check stopping conditions and compare loops A and B:

Stopping conditions:

if either  $|X_i^A(0) - X_{i,given}(0)| < \epsilon$  or  $|X_i^B(0) - X_{i,given}(0)| < \epsilon$ , end and output data

Note: if both are  $< \epsilon$ , select one closest to the target  $X_{i,given}(0)$ .

If the required accuracy is not achieved then one of the following situations applies:

1) If all steps were completed for both loops A and B

a)  $X_i^A(0) > X_{i,given}(0)$  and  $X_i^B(0) < X_{i,given}(0)$ , set  $\delta = \delta/2$  and  $X_e = X_e + \delta$  if  $X_i^A(0)$  was closer to target, or  $X_e = X_e - \delta$  if  $X_i^B(0)$  was closer to target.

Return to Step 2.

b)  $X_i^A(0) > X_{i,given}(0)$  and  $X_i^B(0) > X_{i,given}(0)$ , then  $X_e = X_e - \delta$ . Return to Step 2.

c)  $X_i^A(0) < X_{i,given}(0)$  and  $X_i^B(0) < X_{i,given}(0)$ , then  $X_e = X_e + \delta$ . Return to Step 2.

2) All steps were completed for loops A only

a)  $X_i^A(0) > X_{i,given}(0)$ , set  $\delta = \delta/2$  and  $X_e = X_e + \delta$ . Return to Step 2.

b)  $X_i^A(0) < X_{i,given}(0)$ , set  $\delta = \delta/2$  and  $X_e = X_e + 2\delta$ . Return to Step 2.

3) All steps were completed for loops B only

a)  $X_i^B(0) > X_{i,given}(0)$ , set  $\delta = \delta/2$  and  $X_e = X_e - \delta$ . Return to Step 2.

b)  $X_i^A(0) < X_{i,given}(0)$ , set  $\delta = \delta/2$  and  $X_e = X_e - 2\delta$ . Return to Step 2.

4) Steps were incomplete for both loops

If  $t_s^A \geq t_s^B$ , set  $X_e = X_e + \delta$  and return to Step 2.

Otherwise set  $X_e = X_e - \delta$  and return to Step 2.

### **Step 8**

Continue until solution converges or maximum number of iterations.

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