Competition in Online Markets with Auctions and Posted Prices*

Alexander Maslov†

October 25, 2019

Abstract

In this note I study an online consumer-to-consumer market (e.g. eBay) with limited supply. Sellers may list their items by posted prices or auctions. I show that when there is competition among sellers, they use only posted prices in the equilibrium. This result contrasts with the findings for a monopolistic seller listing objects by auctions and posted prices on markets with infinite supply, where using both mechanisms is the equilibrium. The model helps to explain the trends documented in Einav et al. (2018).

Keywords: E-commerce; Auctions; Price Posting; Competing Mechanism; Duopoly
JEL Codes: D40, D43, D44, L10

1 Introduction

Last decades have witnessed proliferation of the Internet and rampant development of its associated technologies. One of the most salient examples of such development is e-commerce. Internet sales have not only changed the structure of certain retail markets (e.g. recent closure of Sears and significant commercial space reduction by other retailers), but also gave rise to new technological companies posing as catalysts of online trade (e.g. Shopify or Volusion). Of course, the primary focus was on large online trading platforms like eBay, Amazon, Alibaba and others. Online trading significantly reduces buyers' searching costs and facilitates implementation of auctions as listing mechanisms (along with accustomed posted prices). However, despite conspicuous benefits, many platforms in online retail chose not to adopt them. Additionally, platforms that have adopted auctions have seen a steady

*I would like to thank Radovan Vadovič, Aggey Semenov, Sergei Severinov, Joe Harrington, Dave Amundsen, Zhiqi Chen, Andrew Sweeting, Sergiy Pysarenko and especially Jesse Schwartz and René Kirkegaard as well as participants of numerous seminars and conferences for their valuable suggestions and comments. All mistakes are my own.

†Department of Economics, Finance and Quantitative Analysis, Kennesaw State University, 560 Parliament Garden Way, Kennesaw, GA, 30144, e-mail: alexander.maslov@kennesaw.edu

decline in their use over the last decade.[1] This trend was documented in (Einav et al. (2018)). The authors argue that it is more convenient to sell goods by posted prices, and after having experimented with the auctions, many sellers resorted to simple price posting. The analysis in this paper suggests that the decline of online auctions may also be associated with the increased competition on the corresponding markets.

E-commerce is usually subdivided into three segments: business-to-business (B-2-B), business-to-consumer (B-2-C) and consumer-to-consumer (C-2-C). It is interesting that online platforms operating in B-2-B or B-2-C segments normally do not allow sellers to use auctions (e.g. Amazon, Alibaba, DHgate, Etsy etc...), whereas C-2-C platforms — do (e.g. eBay, Listia, Catawiki). The crucial difference between these markets is the structure of supply. Take Amazon, for instance. Businesses may sell their goods either via Amazon or to Amazon, which sells them to final consumers later. Amazon requires businesses to sustain uninterrupted supply and timely delivery, which guarantees the coverage of potentially any demand. The situation is different for eBay comprising mostly consumers selling their goods to other consumers. Individuals usually have a limited number of unique items for sale. At times, it is hard to find identical products, because they differ in quality and depreciation. This paper examines the latter market and describes the behavior of buyers when faced with a limited number of goods sold by posted prices and auctions simultaneously. It further shows that in the equilibrium sellers choose to use posted prices over auctions.

Unlike standard single-good auction models, which received much attention in the past decades, a competing mechanism introduces many sellers competing for buyers. Up until recently, the mechanisms were studied separately, i.e. competition in prices and competition in auctions. In other words, sellers could use only auctions or posted prices, but not both at the same time. Competition in prices has been in the literature for a while and originates primarily from the work on Bertrand competition with capacity constraints, e.g. Levitan and Shubik (1972), Kreps and Scheinkman (1983), Davidson and Deneckere (1986), Osborne and Pitchik (1986), Vives (1986). This literature establishes the existence of support for prices used in the mixed strategy equilibria. Competition in sealed-bid and online auctions has been a more recent phenomenon and was analyzed, among others, by McAfee (1993), Peters and Severinov (1997), Peters and Severinov (2006), Burguet and Sákovics (1999), Virág (2010), where the authors prove that even under mild assumptions, competition drives reserve prices to sellers' marginal costs.

A natural extension of the theoretical analysis of the selling mechanisms was to reunite them, i.e. allow sellers to choose between auctions and posted prices, but the models had to be simplified substantially and / or solved numerically. This paper is no exception. There is limited theoretical literature examining simultaneous use of auctions and posted prices. Etzion et al. (2006), Caldentey and Vulcano (2007) and Sun (2008) model a decision of a buyer arriving to a market between buying an object at a posted price and participating in a second-price sealed-bid auction. They establish a threshold-type buyer based on arrival time and show that buyers with higher valuations buy the good at the posted price, while those with lower valuations — participate in the auction. The authors also prove that using mixed mechanism (auctions and posted prices simultaneously) increases the seller's profits

[1] Originally, Amazon allowed sellers to use auctions, but retired them in 2002.

by providing a form of a third-degree price discrimination between high- and low-valuation buyers.

The aforementioned literature assumes that the seller offers a sealed-bid auction, which is not likely the case in reality. The authors also rely on heuristics of bidders' behavior and numerical simulations to calculate the seller's revenues. Etzion and Moore (2013) analyze the case when a seller offers objects at posted prices and open-bid auctions. The bidders are locked into the auction, but are able to observe their bids and even revise them as the auction continues. The results of the analysis are similar to the findings of the literature exploiting sealed-bid auctions — using dual channel increases the seller's profits. Finally, Hummel (2015) considers a model when an infinite number of objects is sold simultaneously by auctions and posted prices and also shows that in the equilibrium there exists a threshold-type buyer separating buyers who choose to buy the good at the posted price and buyers who choose to participate in the auction. In his model, the seller prefers to use a mixed mechanism, offering some goods by auctions and other goods — by posted prices. It allows the seller to efficiently price discriminate between high- and low-valuation buyers increasing the revenues.

Thus, the literature on the simultaneous use of auctions and posted prices considers a situation when a buyer with unitary demand arrives to a market and faces an effectively infinite number of objects. Some of them are listed by auctions, others — by posted prices. In this context, the buyer's behavior is primarily defined by two parameters: his valuation for the object and the time of arrival to the market relative to the start of the auction. If the buyer values the object high and arrives to the market late, he is more likely to buy it at the posted price in comparison to a low-valuation buyer who arrives early. The intuition is straightforward: buyers' utility is negatively correlated with the time it takes to obtain the object (in reality auctions may run for several days). On the other hand, if a buyer's valuation is less than the posted price, then the best response is to bid his valuation (Vickrey (1961)).

The discussed papers do not consider dynamic bidding, so a buyer has to commit to either of the mechanisms in the beginning. In addition, their framework overlooks that in reality B-2-B and B-2-C online platforms with effectively infinite supply of goods do not allow sellers to use auctions (e.g. Amazon, Rakuten or Alibaba). Finally, only one seller is considered. Here, I do not attempt to characterize the equilibrium in general terms (which seems to be technically impossible), but rather provide a simple example of competition between sellers on online C-2-C markets that contradicts established results for markets with auctions and posted prices. Specifically, when there is competition between sellers, posted prices prevail over the mixed mechanism in the equilibrium.

How does limited supply change buyers' behavior? If there are more buyers than products on a market, some of the buyers may be left without it. Hence, on markets with limited supply, where goods are sold simultaneously by auctions and posted prices, a high-valuation buyer has an incentive to exit the auction early and buy the good at the posted price. If he waits until the end of the auction, the posted price outside may disappear, and even if the auction is won, he may end up paying more than the initial posted price of the good. The reader may have noticed that the incentives above are not unique to this setup. Etzion and Moore (2013) have already pointed out toward similarities between simultaneous use of auctions and posted prices and auctions with a buy-it-now (BIN) price. However, they conclude that "existing literature on auctions with a buy-now price cannot predict the performance of the suggested dual channel strategy when considering a seller with effectively unlimited supply". Maslov (2019) compares buyers' behavior between the models and discusses their common features and differences.

When implementing the BIN option a seller adds a posted price to an auction-style listing allowing buyers to either buy the item right away or place a bid in the auction. On eBay, it is possible to buy the good at the BIN price before the auction starts or until the auction price reaches a "secret" price set by the seller. Yahoo! design allows to buy the good at the BIN price during the auction[2]. When there is one seller, Anwar and Zheng (2015) show that the presence of a BIN price increases his expected revenues and reduces the allocative inefficiency. For many sellers, the authors conclude that between two mechanisms of "auction only" and "auction with a BIN price" the latter is always preferred.

Reynolds and Wooders (2009) prove the existence of a cutoff strategy for both eBay and Yahoo! auctions and show that when bidders are risk averse, introducing the BIN price raises sellers' revenues. Whereas the authors analyze the case of one seller and one object, this article considers a market with two objects offered by competing sellers, which introduces certain alterations to the authors' environment. For example, buying the good at the BIN price would end the auction, but if a bidder leaves the auction and buys the good at the posted price, the auction continues. The buyer who left the auction may have the highest valuation, but he could end up paying more than the other buyer who won the auction. Kirkegaard and Overgaard (2008) examine a somewhat similar case of inefficiency arising in sequential auctions. The existing literature provides different rationale for using the BIN price, but it all documents its positive effect on the seller's profits (e.g. Mathews and Katzman (2006), Bauner (2015), Chen et al. (2013)).

This paper shows how the incentives from auctions with a BIN price shape buyers' behavior on dual-mechanism markets with limited supply. It further analyzes sellers' profits in such environment and compares them to the profits received from other selling mechanisms, i.e. auctions or posted prices only, finding that sellers list their items by posted prices in the equilibrium. An attempt to endogenize the choice of the sellers' mechanisms was done in Julien et al. (2002), where authors conclude that sellers use auctions with reserve prices in the equilibrium. However, their model is quite restrictive, because buyers have the same valuation, which is also public information. From the extensive literature we know that asymmetric information is pivotal in the design of auctions, and their application is heavily distorted when buyers' valuations are common knowledge. Lastly, analyzing listing strategies of a monopolistic seller on C-2-C markets, Maslov (2020) shows that in the equilibrium the seller uses auctions if buyers are risk neutral, and auctions and posted prices if buyers are risk averse.

The following section describes the model. Sections 3, 4 and 5 characterize equilibria for the following subgames: auction - posted price, posted prices only and auctions only. Section 6 discusses the equilibrium of the whole game, and section 7 concludes the paper.

[2] In 2000s Yahoo! discontinued its services in most of the countries except Japan and Taiwan.

2 The Model

Consider two sellers with identical objects (one per seller). They sequentially arrive to an online market and choose whether to list their objects by a posted price or an ascending clock auction without reserve. To simplify things further, assume that the time between their arrival is small enough so that all buyers arrive only after both objects were posted. The buyers have unitary demands and independent private values v_i that are uniformly distributed according to a cumulative distribution function U with support $[0, 1]$. Buyers are able to observe posted prices and the number of active bidders in the auction if at least one of the sellers listed his item by an auction.[3] There is no resale, and sellers' marginal costs are zero.

In what follows I divide the analysis into three subgames providing additional model parameters. After characterizing buyers' strategies and sellers' profits in each of the subgames, I combine the results and analyze the whole game.

3 Auction — Posted Price

3.1 Buyers' Strategies

Consider a situation when one of the items is posted by an auction and the other item is listed by a posted price p, with $0 < p < 1$. Buyers who start bidding in the auction may exit at any time and purchase the item outside. If several buyers decide to leave the auction and buy the good outside (either at the start of the auction or while it is still running), then the good is awarded randomly to one of the buyers, and the rest return to the auction.

Clearly, a buyer with valuation $v \in [0, p]$ cannot afford the object outside, and his best response is to bid his valuation (Vickrey (1961)). This buyer drops out from the auction when the clock moves past his valuation. Buyers with valuations higher than the posted price never have an incentive to drop out from the auction. On the other hand, bidding their valuations cannot be the best response as well. To see this, consider three buyers with $v_i > p$ bidding in the auction. When the clock reaches p each of them would want to exit and buy the good at p, but only one is randomly permitted to do so. The remaining buyers continue bidding in the auction and eventually receive a lower payoff from either having to pay more (since $v_i > p$) or losing the auction. Any buyer can do better by exiting the auction right before the clock matches the price outside. However, forseeing this, other buyers may exit even earlier. Hence, there is such a price $c \in [0, p)$ at which a buyer exits the auction and buys the good at the posted price outside, and there is no loss of generality in looking for an equilibrium in cutoff (exit) strategies when valuations are higher than the posted price outside.

A strategy for a buyer with valuation $v \in (p, 1]$, who comes to the market (at the start of the auction) is a function $z(v)$, which gives each $v \in (p, 1]$ a cutoff (exit) price $c \in [0, p)$ at which the buyer leaves the auction and buys the good at p. The strategy is defined similarly

[3] Maslov (2019) shows that this is a necessary condition for the equilibrium in cutoff strategies to exist in auctions with an outside option.

to auctions with a BIN price. However, if one of the opponents dropped out before c, there are only two buyers left in the auction, and the object is still available at the posted price outside. Clearly, it is no longer profit maximizing to follow strategy $z(v)$. Recall that buyers are able to observe the number of active bidders in the auction. Hence, the remaining buyers change their strategy to $max[v_i, p]$. It is readily seen that when there are only two buyers left in the market and both objects are available, there is neither incentive to exit the auction before p, nor to bid higher than p. If another buyer exited the auction and bought the good at the posted price before c, the remaining buyers simply bid their valuations (Vickrey (1961).

Without loss of generality, consider a buyer with valuation $v \in (p, 1]$. His expected payoff depends on the realization of the valuations of his opponents. Call them x and y with $x > y$. In other words, x is the first order statistic among his opponents, and y is the second order statistic. Then, their joint density is $g(x, y) = 2$. A buyer with valuation v chooses a cutoff price c to maximize his expected payoff (detailed discussion of the objective function is delegated to the Appendix):

$$\max_{c}\Bigg(\int_0^c\int_0^x (v-x)2dydx + \int_c^p\int_0^c (v-x)2dydx + \int_c^p\int_c^x (v-p)2dydx + \int_p^{z^{-1}(c)}\int_0^x (v-p)2dydx + \int_{z^{-1}(c)}^v\int_0^{z(x)} (v-p)2dydx + \int_{z^{-1}(c)}^v\int_{z(x)}^x (v-y)2dydx + \int_v^1\int_0^{z(x)} (v-p)2dydx + \int_v^1\int_{z(x)}^v (v-y)2dydx\Bigg) \qquad (1)$$

Let $\hat{v}$ denote a buyer who is indifferent between bidding in the auction and buying the good outside and let $\tilde{p}$ be a threshold price beyond which the indifferent buyer does not exist. Then:

Lemma 1. *For all $v > p$ and $p > \tilde{p}$ the cutoff (exit) function $z(v)$ is a monotonic and decreasing function: $z(v) = \frac{7}{4}p - \frac{3}{4}v$. For $v > p$ and $p \leq \tilde{p}$ the equilibrium exit function is discontinuous at $z(v) = \hat{v}$.*

Proof. Appendix.

Function $z(v)$ maps buyers' valuations into corresponding exit prices. It depends on the buyer's valuation and the posted price of the outside object. If the posted price is high enough, then there are no buyers who want to buy the good outside at the beginning of the auction, and the function is strictly continuous. At lower posted prices, however, buyers with higher valuations prefer not to bid in the auction, but rather buy the good at the posted price right away. The discontinuity occurs at the valuation of a buyer, who is indifferent between following strategy $z(\cdot)$ and leaving the auction at the beginning in attempt to buy the object at the posted price p. All buyers with lower valuations (but higher than the posted price) submit bids according to $z(\cdot)$. Buyers with higher valuations attempt to buy the good at the posted price before the auction starts. Only one of them succeeds, and the rest return to the auction.

Proposition 1. *There is an equilibrium in which:*

a) Buyers with valuations less than the posted price bid their valuations.

b) When an indifferent buyer does not exist ($\tilde{p} > \frac{3}{5}$), buyers with valuations higher than the posted price follow a cutoff strategy prescribed by $z(v) = \frac{7}{4}p - \frac{3}{4}v$.

c) When an indifferent buyer exists ($\tilde{p} < \frac{3}{5}$), buyers with valuations higher than the posted price, but lower than the valuation of the indifferent buyer ($\hat{v} = \frac{5}{3}p$) follow a cutoff strategy prescribed by $z(v) = \frac{7}{4}p - \frac{3}{4}$.

d) When an indifferent buyer exists ($\tilde{p} < \frac{3}{5}$), buyers with valuations higher than the valuation of the indifferent buyer ($\hat{v} = \frac{5}{3}p$), exit the auction at the beginning and attempt to buy the object at the posted price outside.

Proof. If $v < p$ then in the ascending clock auction the best response is to bid one's valuation (Vickrey (1961)). From Lemma 1 it follows that when $v > p$ and $p > \tilde{p}$ the equilibrium is completely determined by $z(v)$. To find the value of the indifferent buyer when $p \leq \tilde{p}$, set equal the expected payoff a buyer gets from following strategy $z(v)$ and the expected payoff that he receives from exiting the auction at the beginning:

$$\int_{\hat{v}}^{1}\int_{0}^{\hat{v}}\frac{1}{2}(\hat{v}-p+\hat{v}-y)2dydx+\int_{\hat{v}}^{1}\int_{\hat{v}}^{x}\frac{1}{3}(\hat{v}-p)2dydx+\int_{0}^{\hat{v}}\int_{0}^{x}(\hat{v}-p)2dydx= \\ \int_{0}^{c}\int_{0}^{x}(\hat{v}-x)2dydx+\int_{c}^{p}\int_{0}^{c}(\hat{v}-x)2dydx+\int_{c}^{p}\int_{c}^{x}(\hat{v}-p)2dydx+ \\ \int_{p}^{\hat{v}}\int_{0}^{x}(\hat{v}-p)2dydx+\int_{\hat{v}}^{1}\int_{0}^{\hat{v}}(\hat{v}-y)2dydx \tag{2}$$

where $c = \frac{7}{4}p - \frac{3}{4}\hat{v}$.

The LHS shows the expected payoff of the buyer if he exits the auction at the beginning and attempts to buy the object outside. The RHS denotes the expected payoff of the buyer if he follows the cutoff strategy. It can further be simplified to:

$$64p + 91p^3 + (64p - 64 - 81p^2)v + (32 - 47p)v^2 + 5v^3 = 0 \tag{3}$$

The solution to (3) would uniquely define $\hat{v}(p) \in (p, 1]$. Unfortunately, there are no "clean" solutions to this equation. Chen et al. (2013) have encountered a similar problem and assumed that buyers with high valuations move first. That would put the indifferent buyer at the intersection of the exit function and the v axis. Their reasoning revolves around justifying the exogenous priority of arrival of buyers with higher valuations.[4] I believe it is a very strong assumption in the world of asymmetric information and suggest the following alternative. We know that $\hat{v} \in (p, z^{-1}(0))$, and the indifferent buyer may be anywhere within this support. Let us assume that the value of the indifferent buyer is the average between

[4]See Chen et al. (2013) for details.

the infimum and the supremum of this support, i.e. $\hat{v} = \frac{p+z^{-1}(0))}{2} = \frac{p+7/3p}{2} = \frac{5}{3}p$. Then, it follows that $z(v|v \leq \frac{5}{3}p) = z(v)$ and $z(v|v > \frac{5}{3}p) = 0$. To find the price at which the indifferent buyer does not exist, simply set $\frac{5}{3}\tilde{p} = 1$ to receive $\tilde{p} = \frac{3}{5}$. The rest of the proof follows directly from Lemma 1. □

3.2 Sellers' Revenues

Recall that sellers arrive sequentially to the market, and in this subgame one of them chooses a posted price, and another one — an auction. Previously found function $z(\cdot)$ is useful in determining sellers' profits when goods are sold simultaneously by an auction and a posted price. Sellers face three buyers. Let x_1, x_2 and x_3 be the first (highest), second and third order statistics of their valuations correspondingly. Then, their joint density is $\phi(x_1, x_2, x_3) = 6$. A seller needs to account for all possible realizations of buyers' valuations, but it has to be done for two distinct scenarios — when an indifferent buyer exists and when he does not. The reason for that is the additional space comprising buyers with valuations higher than the valuation of the indifferent buyer. When the latter does not exist, this space is an empty set, and the seller does not account for it in the expected profit.

Lemma 2.
a) If an indifferent buyer exists, then the profits of sellers from using an auction and a posted price are described by the following functions:

$$\begin{cases} \pi_A(p) = \frac{1}{3} - \frac{5p}{9} + \frac{25p^2}{18} - \frac{125p^3}{81} + \frac{8251p^4}{3888} \\ \pi_P(p) = p - \frac{17p^4}{6} \end{cases} \quad \text{where } p \leq \frac{3}{5} \tag{4}$$

b) If an indifferent buyer does not exist, then the profits of sellers from using an auction and a posted price are described by the following functions:

$$\begin{cases} \pi_A(p) = -\frac{71}{256} + \frac{135p}{64} - \frac{213p^2}{128} + \frac{7p^3}{64} + \frac{57p^4}{256} \\ \pi_P(p) = \frac{25p}{16} - \frac{27p^2}{16} - \frac{21p^3}{16} + \frac{23p^4}{16} \end{cases} \quad \text{where } p > \frac{3}{5} \tag{5}$$

Proof. Appendix

The profit functions above describe a continuum of profits for both sellers when they use different mechanisms to sell their items. Depending on the posted price chosen by one seller, the profit of the seller using the auction is defined by one of the π_A equations.

Proposition 2. *When one seller lists his item by an auction and another seller uses a posted price, the best response of the seller with the posted price is to set it optimally. The seller who uses the auction earns* 0.308*, and the seller selling by the posted price earns* 0.334*.*

Proof. Consider a first-moving seller who lists his item by an auction (w.l.o.g). If the other seller can respond only with a posted price, then he will set it at a profit-maximizing level by the assumption of rationality. The optimal posted price is found by solving $\frac{d\pi_P}{dp} = 0$. The profit of the seller who uses the auction is found by substituting the profit-maximizing posted price of the π_P profit stream into the π_A function:

$$\frac{d\pi_P}{dp} = 1 - \frac{34p^3}{3} = 0, \text{ where } p \leq \frac{3}{5} \tag{6}$$

$$\frac{d\pi_P}{dp} = \frac{25}{16} - \frac{27p}{8} - \frac{63p^2}{16} + \frac{23p^3}{4} = 0, \text{ where } p > \frac{3}{5} \tag{7}$$

Equation (6) holds when $p = 0.445$; equation (7) — when $p = 0.387$, which is outside the defined range for p. Hence, $p^* = 0.445$ is the price chosen by one seller as the response to the other seller using the auction. Substituting it into the corresponding profit functions (i.e. when $p \leq \frac{3}{5}$) for both sellers produces:

$$\pi_A(p^*) = \frac{1}{3} - \frac{5 \times 0.445}{9} + \frac{25 \times (0.445)^2}{18} - \frac{125 \times (0.445)^3}{81} + \frac{8251 \times (0.445)^4}{3888} = 0.308 \tag{8}$$

$$\pi_P(p^*) = 0.445 - \frac{17 \times (0.445)^4}{6} = 0.334 \tag{9}$$

where π_A is the profit of the seller who uses the auction and π_P is the profit of the seller who uses the posted price. □

Even when buyers are risk neutral, a seller who responds with a posted price earns more than the seller with the auction. The next corollary shows that the profit of the former seller goes up even further if buyers are risk averse.

Corollary 1. *When one seller uses an auction and the other seller uses a posted price to sell their items, the profit of the last seller increases with buyers' risk aversion.*

Proof. Corollary 3 from Reynolds and Wooders (2009) shows that an increase in risk aversion shifts the buyers' exit function down (left) making the slope steeper. This result is intuitive: it indicates that risk-averse buyers are less likely to wait in the auction and prefer to buy the good at the BIN price sooner than their risk-neutral counterparts. Due to similarities between the exit functions, risk aversion affects the exit function in this model the same way. Using Lemma 2, it is easy to check that increasing the slope of the exit function results in a higher profit for the seller with the posted price and a lower profit for the seller with the auction. For example, for a twice steeper exit function the profits change to $\pi_A = 0.298$ and $\pi_P = 0.380$. □

4 Posted Price — Posted Price

When two sellers compete in posted prices they play a game similar to Bertrand competition with homogeneous products and capacity constraints. A detailed exposition of this game can be found in, for example, Levitan and Shubik (1972) or Kreps and Scheinkman

(1983). However, the model here is developed for the realm of asymmetric information, which introduces certain alterations to the demand. In addition, sellers have fixed capacity and move sequentially.

4.1 Buyers' Strategies

When presented with a choice between two identical goods listed by different posted prices, a buyer will always choose the lower-priced item. If objects are listed at the same price, buyers would be indifferent between them. When supply is finite, the rationing rule matters. The literature on Bertrand competition with capacity constraints has established the following ways to split the demand: "surplus-maximizing" rationing scheme (e.g. Kreps and Scheinkman (1983), Osborne and Pitchik (1986)), when buyers with higher valuations have "priority" in choosing between the objects, and "proportional" rationing scheme (e.g. Allen and Hellwig (1986), Dasgupta and Maskin (1986)), when objects are awarded randomly to buyers. The former rule would not work well in the world of asymmetric information. Hence, I follow the proportional scheme, when each buyer with a valuation higher than the posted price is equally likely to get the object.

4.2 Sellers' Revenues

When both objects are sold by posted prices, it is straightforward to calculate sellers' expected profits knowing the support of the distribution of buyers' valuations and the rationing rule. Recall that x_1 is the first order statistic among buyers' valuations from the seller's perspective, x_2 - second order statistic and x_3 - third order statistic.

Proposition 3. *When both sellers list their items by posted prices, one of them sets an ex-post low price $p_l = 0.325$ and the other seller chooses an ex-post high price $p_h = 0.565$. Both sellers earn the same profit: $\pi_{p_l} = \pi_{p_h} = 0.314$.*

Proof. A seller who sells his item at an ex-post low price will always be able to sell it if at least the first order statistic of the buyers' valuations is above this price:

$$\pi_{p_l}(p_l) = 6\int_{p_l}^{1} p_l f(x_1)dx_1 = p_l - p_l^4 \qquad (10)$$

where $f(x_1)$ is the marginal density of the first order statistic.

The seller with an ex-post high posted price will be able to sell his item if one buyer's valuation is above it and this buyer fails to buy at a lower price or if at least two buyers' valuations are above this price. This may happen in several ways:

a) $p_h < x_1 < 1$, $p_l < x_2 < p_h$ and $0 < x_3 < p_l$. In this case, buyers x_1 and x_2 first attempt to buy the object at the lower price. With 1/2 probability the first order statistic fails to buy it at p_l and instead buys at p_h: $\int_{p_h}^{1}\int_{p_l}^{p_h}\int_{0}^{p_l}\frac{1}{2}p_h 6dx_3dx_2dx_1$.

b) $p_h < x_1 < 1$, $p_l < x_2 < p_h$ and $p_l < x_3 < x_2$. In this case, all buyers first attempt to buy the object at the lower price. With probability 2/3 the first order statistic fails to buy it at p_l and instead buys at p_h: $\int_{p_h}^{1}\int_{p_l}^{p_h}\int_{p_l}^{x_2}\frac{2}{3}p_h 6dx_3dx_2dx_1$.

c) $x_1, x_2 > p_h$. In this case, the seller with an ex-post high posted price always sells his item: $\int_{p_h}^{1}\int_{p_h}^{x_1}\int_{0}^{x_2}p_h 6dx_3dx_2dx_1$.

Hence, the profit of the seller selling at an ex-post high posted price is:[5]

$$\pi_{p_h}(p_l,p_h) = 6\Bigg[\int_{p_h}^{1}\int_{p_l}^{p_h}\int_{0}^{p_l}\frac{1}{2}p_h dx_3dx_2dx_1 + \int_{p_h}^{1}\int_{p_l}^{p_h}\int_{p_l}^{x_2}\frac{2}{3}p_h dx_3dx_2dx_1 + \int_{p_h}^{1}\int_{p_h}^{x_1}\int_{0}^{x_2}p_h dx_3dx_2dx_1\Bigg] = (p_h-1)p_h(p_l-1)(1+p_l+p_h) \tag{11}$$

First, observe that $\pi_{p_l}(p_l)$ is a singe-peaked function, because it is a product of increasing and decreasing continuous functions p_l and $1-p_l^3$. Hence, $\frac{d\pi_{p_l}(p_l)}{dp_l} > 0|p_l < p_l^*$ and $\frac{d\pi_{p_l}(p_l)}{dp_l} < 0|p_l > p_l^*$. Function $\pi_{p_l}(p_l)$ reaches its maximum when $\frac{d\pi_{p_l}(p_l)}{dp_l} = 0$ or at $p_l^* = 0.623$.

Second, $\pi_{p_h}(p_l,p_h)$ is also a single-peaked function by the same argument (the product of two increasing and two decreasing continuous functions). However, the maximum is conditional on the ex-post low price. It is easy to check with numerical simulations that $\forall p_l \implies p_h^* \in (0.55, 0.58)$. Further, $\frac{\partial\pi_{p_h}(p_l,p_h)}{\partial p_l} = (p_h-1)p_h(p_h+2p_l) < 0\ \forall p_l$ meaning that the seller with an ex-post high price always prefers ex-post low price to be as low as possible.

Next, to prove that $\forall p$ a seller with $p-\epsilon$ earns more profit, consider the profit function of the seller with an ex-post high price p:

$$\begin{aligned}\pi_{p_h}(p,p) &= p(1-p)^2(1+2p) = p(p^2-2p+1)(1+2p)\\ &= p^3(1+2p) + p(1-4p^2) = 2p^4 - 3p^3 + p\end{aligned}$$

Subtract this profit from the profit of the seller with an ex-post low price ($p = p-\epsilon$):

$$\pi_{p_l}(p) - \pi_{p_h}(p,p) = (p-p^4) - (2p^4-3p^3+p) = -3p^4+3p^3 = 3p^3(1-p) \geq 0\ \forall p$$

The last equation implies that $\pi_{p_l}(p) - \pi_{p_h}(p,p) \geq 0\ \forall p_l = p_h - \epsilon\ \&\ p_h \in (0,1]$. The next step is to apply backward induction. Consider the best response of the second-arriving seller to an arbitrary price p set before him:

a) $p > p_l^*$. It is readily seen that the second-arriving seller sets p_l^*.

b) $p < p_l^*$. The second-arriving seller may either undercut the first seller (setting $p_l = p-\epsilon$) or price at p_h^* depending on what is more profitable. Notice that because $\frac{\partial\pi_{p_h}(p_l,p_h)}{\partial p_l} < 0$ the profit of the seller with an ex-post high price increases with the decrease in the ex-post low price. On the other hand, because $\frac{d\pi_{p_l}(p_l)}{dp_l} > 0$ for $p_l < p_l^*$ the profit of the seller with an ex-post low price falls with the decrease in p_l. Thus, there must be such

[5] Notice that the profit of the seller with an ex-post high price depends on both prices, while the profit of the seller with an ex-post low price — only on low price.

a price $\bar{p}_l \in (0, p_l^*)$, at which sellers' profits coincide, and beyond which it will no longer be profitable to undercut. Hence, the following equality must hold at $\bar{p}_l$:

$$\pi_{\bar{p}_l}(\bar{p}_l) = \pi_{p_h}(\bar{p}_l, p_h^*(\bar{p}_l)) \tag{12}$$

To find $p_h^*(\bar{p}_l)$ first take the derivative of the profit function from an ex-post high price with respect to this price and express it as a function of the ex-post low price:

$$\frac{\partial \pi_{p_h}(\bar{p}_l, p_h)}{\partial p_h} = (\bar{p}_l - 1)(3p_h^2 - 1 + (2p_h - 1)\bar{p}_l) = 0$$
$$\implies p_h^*(\bar{p}_l) = \frac{\bar{p}_l - \bar{p}_l^2 - \sqrt{3 - 3\bar{p}_l - 2\bar{p}_l^2 + \bar{p}_l^3 + \bar{p}_l^4}}{3(\bar{p}_l - 1)}$$

Plugging $p_h^*(\bar{p}_l)$ into (12) and solving for $\bar{p}_l$ results in $\bar{p}_l = 0.325$ with $\pi_{p_l}(\bar{p}_l) = \pi_{p_h}(\bar{p}_l, p_h^*) = 0.314$. Using $\pi_{p_h}(\bar{p}_l, p_h^*) = 0.314$, it is straightforward to find the corresponding ex-post high price $p_h^* = 0.565$.

It easy to check that the first-arriving seller has no incentive to deviate from the equilibrium price $\bar{p}_l$. If he sets a price higher than $\bar{p}_l$, then the second-arriving seller undercuts him and by definition $\pi_{p_h}(p_l, p_l) < \pi_{p_l}(\bar{p}_l)$ for $p_l > \bar{p}_l$. If the former seller sets a price lower that $\bar{p}_l$, then the latter seller sets p_h^*, and earns even higher profit, because $\frac{\partial \pi_{p_h}(p_l, p_h)}{\partial p_l} < 0$. Hence, the first-arriving seller may increase his profit by pricing exactly at $\bar{p}_l$. □

5 Auction - Auction

The behavior of buyers in online auctions was fully characterized by Peters and Severinov (2006). They proved that buyers are cross bidding, and that trades occur at a uniform price equal to the $m+1^{\text{st}}$ highest value among the buyers' valuations and sellers' reserve prices, where m is the number of sellers each selling a unit. In this model, there are no reserve prices, which means that both auctions will end at the value of the third order statistic x_3.[6]

$$\pi_A^1 = \pi_A^2 = \int_0^{\bar{v}} \int_0^{x_1} \int_0^{x_2} x_3 6 dx_3 dx_2 dx_1 = 0.25 \tag{13}$$

6 Discussion

Having found the equilibria for all the subgames I proceed to characterizing the equilibrium of the whole game.

[6]For a detailed analysis of the competition among sellers in online auctions see Maslov and Schwartz (2020).

6.1 First seller chooses an auction

If the first-moving seller chooses an auction, the other seller may reply with either an auction or any posted price $p \in [0, 1]$. If he replies with the auction, the expected profits of the sellers are equal to 0.25. If the latter seller replies with the posted price, then both sellers' revenues are defined by profit functions from Lemma 2. From proposition 2 we already know that in this case the second-moving seller chooses the optimal posted price $p^* = 0.445$, and the sellers' profits are equal to $\pi_A(p^*) = 0.308$ and $\pi_P(p^*) = 0.334$.

6.2 First seller chooses a posted price

The first-arriving seller may choose any price $p \in [0, 1]$, and it can be replied with either an auction or any posted price by the other seller, which results in a continuum of profits. However, I show that the price domain may be subdivided into several regions prompting the same response from the other seller. Figure 1 is helpful in this analysis. It depicts profit functions describing sellers' revenues when one seller replies with either a posted price (depending on whether it is higher or lower than the price of the other seller) or an auction to the other seller's choice of the posted price. There are several possibilities:

a) If the first seller names the price $p = \bar{p}_l = 0.325$, and the second seller replies with the auction, the latter seller earns $\pi_A(\bar{p}_l) = 0.270$, which is below the profit that he earns by replying with the optimal high posted price $p_h^* = 0.565$, i.e. $\pi_{p_h}(\bar{p}_l, p_h^*) = 0.314$.

b) If the first-moving seller's price is below $\bar{p}_l$, it is not profitable for the second seller to undercut, so he would prefer the optimal high posted price p_h^*. However, at some prices $p < \bar{p}_l$ it may be more profitable to reply with an auction. To find the threshold price p_1^t the following must hold: $\pi_A(p_1^t | p_1^t \leq \frac{3}{5}) = \pi_{p_h}(p_1^t, p_h^*(p_1^t))$. Solving for p_1^t produces $p_1^t = -0.09$, which means that it is always profitable to set the optimal ex-post high posted price in response to any $p < \bar{p}_l$.

c) If the first-moving seller names a price higher than $\bar{p}_l$, but below p_l^* it is easy to check that the other seller will undercut (set $p - \epsilon$), because $\pi_{p_l}(p_l) > \pi_A(p)$ and $\pi_{p_l}(p_l) > \pi_{p_h}(p_l, p_h^*)$ $\forall\ p_l \in (\bar{p}_l, p_l^*)$.

d) If the first-moving seller names a price $p > p_l^*$ then at some prices it would be more profitable for the second seller to set p_l^*, while for other prices — reply with an auction. To find the threshold price p_2^t set $\pi_A(p_2^t | p_2^t > \frac{3}{5}) = \pi_{p_l}(p_l^*)$. Solving for p_2^t produces $p_2^t = 0.674$.

Summarizing, $\bar{p}_l = 0.325$, $p_l^* = 0.630$ and $p_2^t = 0.674$ (points A, B and C on Figure 1). If $p \leq \bar{p}_l$ then it is the best response to set the optimal ex-post high price. If $\bar{p}_l < p < p_l^*$ then the best response is to price at $p - \epsilon$. For $p_l^* < p < p_2^t$ the best response is to set the optimal ex-post low price, and for $p > p_2^t$ replying with the auction earns the highest profit.

The response of the second seller remains the same for the analyzed four segments of the profit functions, but the profits of the first seller vary. Plugging in the numbers into the corresponding profit functions we can identify the maximum profit the first seller may earn for each of the segments. When $p < 0.325$, the second seller replies with the optimal high posted price, and the maximum the first seller may earn is by pricing $p = 0.325 - \epsilon$. His profit is then $\pi_{p_l}(p_l) = 0.314 - \epsilon$. When $p \in (0.325, 0.630)$, the second seller undercuts, and the first seller's profit becomes $\pi_{p_h}(p_l, p_h)$. This profit is maximized at $p = 0.325 + \epsilon$ and is equal to $\pi_{p_h}(p,p) = 0.244$. When $p \in (630, 0.674)$, the second seller sets the optimal low posted price, and the first seller's profit becomes $\pi_h(p_l^*, p_h)$. The maximum he may earn is at $p = 0.630 + \epsilon$, which is equal to 0.195. Lastly, when $p > 0.674$, the second seller replies with the auction, and the maximum the first seller may earn is by pricing at $p = 0.674 + \epsilon$, which gives him a profit of 0.181.

6.3 Equilibrium

Based on the analysis above, I can characterize the equilibrium of the whole game:

Proposition 4. When sellers compete in posted prices and auctions, it is optimal for the first-moving seller to list his item by an ex-post low posted price. The second-arriving seller replies with an ex-post high posted price. Both sellers earn the same profits.

Proof. The proof follows directly from the discussion above. When the first seller lists his item by an auction, the second seller replies with the optimal posted price, and the first seller earns 0.308. When the first seller lists his item by a posted price making the second-arriving seller indifferent between undercutting and setting the optimal ex-post high posted price, he earns 0.314. The continuum of payoffs for both players is defined by the following profit functions: π_A, π_P, π_{p_l} and π_{p_h}. When only one of the two sellers uses an auction, their profits are defined by π_A and π_P from Lemma 2. If both sellers use auctions, their profits are equal to 0.25. If the sellers use only posted prices, then their profits are defined by π_{p_l} and π_{p_h} from Proposition 3. Hence, these functions account for all possible off-equilibrium paths. □

Previously, I have shown that a seller may earn more by switching to an auction if, for example, the first seller names a very high price or undercut if the price set by the first seller is somewhere in the middle range. However, foreseeing this, the first seller sets such a posted price that makes the second-arriving seller indifferent between undercutting and setting his optimal posted price. The second-arriving seller earns less if he replies with the auction instead of the posted price. Because both sellers earn the same profit in the equilibrium, there is no mover advantage.

Literature has shown that it is profitable for a monopolistic seller to use different mechanisms in the equilibrium, because they allow to effectively discriminate consumers based on their valuations, time of arrival and risk aversion. The key insight of this environment is that one mechanism generates substantially more profit in comparison to the other mechanism. It engenders infeasibility of applying both mechanism in competitive environment, because one seller will always be at a disadvantage and would have an incentive to deviate.

It is not possible to perform welfare analysis without reserve prices, but market profits (equivalent to the producer surplus in this model) across different mechanisms can be compared. When sellers use only posted prices, in the equilibrium, they earn 0.628. When only auctions — 0.5. Finally, for a mixed mechanism, when one seller lists his item by an auction, and the other one — by a posted price, the profits add up to 0.642. This number is further increased with buyers' risk aversion (according to Corollary 1).

7 Conclusion

The paper studied the choice of a selling mechanism by competing sellers on online C-2-C markets with limited supply. Sellers could list their items either by a posted price or an auction. It showed that among investigated combinations of selling mechanisms, i.e. posted price-posted price, auction-auction and auction-posted price, the former emerges in the equilibrium. Unlike models with a monopolistic seller who uses either auctions or mixes between the mechanisms to list his items, when there is competition between sellers, they both choose posted prices in the equilibrium.

The literature on simultaneous use of auctions and posted prices has shown that a monopolistic seller is able to price discriminate consumers by offering different selling mechanisms. However, if two distinct sellers use those mechanisms, one of the sellers gets the "premium" channel, which generates extra revenue from price discrimination at the cost of the other channel left to the second seller. As a result, the latter seller always has an incentive to deviate. Thus, competition compels sellers to use posted prices, and increasing competition on online platforms during the last decade may help to explain the decline of auctions documented in (Einav et al. (2018)).

Finally, it was shown that sellers are worse off under competition in posted prices in comparison to the mixed mechanism. This difference further increases with buyers' risk aversion, which makes the latter mechanism more profitable.

References

Allen, B. and M. Hellwig (1986). Bertrand-edgeworth oligopoly in large markets. *The Review of Economic Studies 53*(2), 175–204.

Anwar, S. and M. Zheng (2015). Posted price selling and online auctions. *Games and Economic Behavior 90*, 81–92.

Bauner, C. (2015). Mechanism choice and the buy-it-now auction: A structural model of competing buyers and sellers. *International Journal of Industrial Organization 38*, 19–31.

Burguet, R. and J. Sákovics (1999). Imperfect competition in auction designs. *International Economic Review 40*(1), 231–247.

Caldentey, R. and G. Vulcano (2007). Online auction and list price revenue management. *Management Science 53*(5), 795–813.

Chen, J.-R., K.-P. Chen, C.-F. Chou, and C.-I. Huang (2013). A dynamic model of auctions with buy-it-now: Theory and evidence. *The Journal of Industrial Economics 61*(2), 393–429.

Dasgupta, P. and E. Maskin (1986). The existence of equilibrium in discontinuous economic games, i: Theory. *The Review of economic studies 53*(1), 1–26.

Davidson, C. and R. Deneckere (1986). Long-run competition in capacity, short-run competition in price, and the cournot model. *The Rand Journal of Economics*, 404–415.

Einav, L., C. Farronato, J. Levin, and N. Sundaresan (2018). Auctions versus posted prices in online markets. *Journal of Political Economy 126*(1), 178–215.

Etzion, H. and S. Moore (2013). Managing online sales with posted price and open-bid auctions. *Decision Support Systems 54*(3), 1327–1339.

Etzion, H., E. Pinker, and A. Seidmann (2006). Analyzing the simultaneous use of auctions and posted prices for online selling. *Manufacturing & Service Operations Management 8*(1), 68–91.

Hummel, P. (2015). Simultaneous use of auctions and posted prices. *European Economic Review 78*, 269–284.

Julien, B., J. Kennes, and I. King (2002). Auctions beat posted prices in a small market. *Journal of Institutional and Theoretical Economics (JITE)/Zeitschrift für die gesamte Staatswissenschaft*, 548–562.

Kirkegaard, R. and P. B. Overgaard (2008). Buy-out prices in auctions: seller competition and multi-unit demands. *The RAND Journal of Economics 39*(3), 770–789.

Kreps, D. M. and J. A. Scheinkman (1983). Quantity precommitment and bertrand competition yield cournot outcomes. *The Bell Journal of Economics 14*(2), 326–337.

Levitan, R. and M. Shubik (1972). Price duopoly and capacity constraints. *International Economic Review 13*(1), 111–122.

Maslov, A. (2019). A note on buyers' behavior in auctions with an outside option. Working paper.

Maslov, A. (2020). Single versus dual channel in online markets with finite supply. Working paper.

Maslov, A. and J. A. Schwartz (2020). Imperfect competition in online auctions with sequential entry. Working paper.

Mathews, T. and B. Katzman (2006). The role of varying risk attitudes in an auction with a buyout option. *Economic Theory 27*(3), 597–613.

McAfee, R. P. (1993). Mechanism design by competing sellers. *Econometrica: Journal of the econometric society 61*(6), 1281–1312.

Osborne, M. J. and C. Pitchik (1986). Price competition in a capacity-constrained duopoly. *Journal of Economic Theory 38*(2), 238–260.

Peters, M. and S. Severinov (1997). Competition among sellers who offer auctions instead of prices. *Journal of Economic Theory 75*(1), 141–179.

Peters, M. and S. Severinov (2006). Internet auctions with many traders. *Journal of Economic Theory 130*(1), 220–245.

Reynolds, S. S. and J. Wooders (2009). Auctions with a buy price. *Economic Theory 38*(1), 9–39.

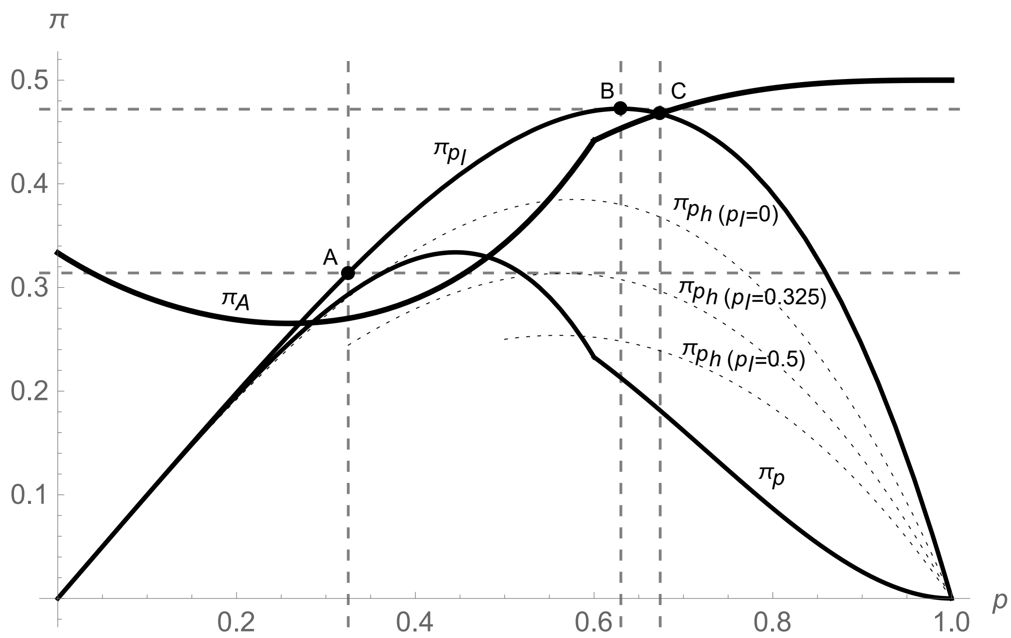
Sun, D. (2008). Dual mechanism for an online retailer. *European Journal of Operational Research 187*(3), 903–921.

Vickrey, W. (1961). Counterspeculation, auctions, and competitive sealed tenders. *The Journal of finance 16*(1), 8–37.

Virág, G. (2010). Competing auctions: finite markets and convergence. *Theoretical Economics 5*(2), 241–274.

Vives, X. (1986). Rationing rules and bertrand-edgeworth equilibria in large markets. *Economics Letters 21*(2), 113–116.

Figure 1: The continuum of sellers' profits when the first seller chooses a posted price.[7]



[7]Notice that when one seller uses a posted price, and the other seller replies with an auction, their profit functions have a kink at $p = \frac{3}{5}$.

A Proof of Lemma 1

Without loss of generality, consider a buyer with valuation v. If x is the first order statistic among his opponents, and y is the second order statistic, then there are several possibilities for their realizations:[8]

a) $x, y < c$. In this situation, x and y cannot do better but to bid their valuations. Both of the opponents drop out when the auction price reaches their valuations, and the buyer in question wins the auction paying the second highest bid: $\int_0^c \int_0^x (v-x)2dydx$.

b) (i) $c < x < p$ and $y < c$. In this scenario, when the auction price reaches y, this buyer drops out. At c, the buyer with valuation v observes that one of his opponents has dropped out from the auction and that the good outside is still available, so he no longer exits the auction. It is easy to check that his best response is to bid $max[v_i, p]$ instead. Because $x < p$, the first order statistic stays in the auction until its price reaches the value of x. Hence, the buyer in question pays the second highest bid: $\int_c^p \int_0^c (v-x)2dydx$.

(ii) $c < x < p$ and $c < y < x$. In this case, when the auction price reaches the exit price of the buyer in question, he leaves the auction and buys the good at the posted price: $\int_c^p \int_c^x (v-p)2dydx$.

c) (i) $p < x < z^{-1}(c)$ and $y < c$. This situation is similar to the one described in b(i): the auction continues until the auction price reaches y: second order statistic drops out from the auction, and the remaining bidders play according to $max[v_i, p]$. Bidding continues until the auction price matches the outside posted price. At p, one bidder leaves the auction and buys the good at the posted price, while the other buyer wins the auction and pays the same amount: $\int_p^{z^{-1}(c)} \int_0^c (v-p)2dydx$.

(ii) $p < x < z^{-1}(c)$ and $c < y < p$. In this case, when the auction price reaches c, the buyer in question leaves the auction and buys the good at the posted price: $\int_p^{z^{-1}(c)} \int_c^p (v-p)2dydx$.

(iii) $p < x < z^{-1}(c)$ and $p < y < x$. Nothing changes from c(ii) — buyer in question is still the first to leave the auction and buy the good at the posted price: $\int_p^{z^{-1}(c)} \int_p^x (v-p)2dydx$.[9]

d) (i) $z^{-1}(c) < x < v$ and $y < z(x)$. In this case, y drops out before the exit price of the first order statistic, and the remaining buyers (v and x) bid according to $max[v_i, p]$. The auction continues until its price matches the outside posted price. At p one bidder leaves the auction and buys the good at the posted price, while the other buyer wins the auction and pays the same amount: $\int_{z^{-1}(c)}^v \int_0^{z(x)} (v-p)2dydx$.

(ii) $z^{-1}(c) < x < v$ and $z(x) < y < x$. In this scenario, the first order statistic leaves the auction and buys the good at the posted price. The remaining buyers bid according to $max[v_i, p]$. When the auction price reaches y, the buyer in question wins the auction and receives a payoff of $v - y$. Since the good outside of the auction is no longer available it is

[8] Here we consider a situation when the true valuation of the first buyer is greater than his "reported value", i.e. $v > z^{-1}(c)$. It is easy to show that when $v < z^{-1}(c)$ the domains of integration remain unchanged, because $\int_p^v \int_0^x (v-p)2dydx + \int_v^{z^{-1}(c)} \int_0^x (v-p)2dydx = \int_p^{z^{-1}(c)} \int_0^x (v-p)2dydx$.

[9] Note that c(i), c(ii) and c(iii) can be summed up to: $\int_p^{z^{-1}(c)} \int_0^x (v-p)2dydx$.

readily seen that the same outcome will also be observed when $c < y < x$: $\int_{z^{-1}(c)}^{v} \int_{z(x)}^{x} (v - y)2dydx$.

e) (i) $v < x < 1$ and $y < z(x)$. In this case, the second order statistic drops out from the auction before the exit price of the first order statistic, and the remaining buyers bid according to $max[v_i, p]$, resulting in both buyers getting the objects at p: $\int_{v}^{1} \int_{0}^{z(x)} (v-p)2dydx$

(ii) $v < x < 1$ and $z(x) < y < v$. In this scenario, the first order statistic leaves the auction and buys the good at the posted price. The remaining buyers bid according to $max[v_i, p]$. When the auction price reaches y, the buyer in question wins the auction and receives a payoff of $v - y$. Since the good outside of the auction is no longer available, it is readily seen that the same outcome will also be observed when $c < y < v$: $\int_{v}^{1} \int_{z(x)}^{v} (v - y)2dydx$.

(iii) When $x, y > v$ the buyer in question gets a payoff of 0.

Summing up the preceding integrals and simplifying the resulting maximization problem produces the following differential equation:

$$(p - z(v))^2 + \frac{(v - z(v))(v + z(v) - 2p)}{z'(v)} = 0 \tag{14}$$

Rewrite equation (14) as follows (to simplify notation use z instead of $z(v)$, also note that $(p - z)^2 = (z - p)^2$):

$$(z - p)^2 z' + (v - z)(v + z - 2p) = 0 \tag{15}$$

Substituting $Z = z - p$ and $V = v - p$ gives:

$$\begin{gathered} Z^2 Z' + (V - Z)(V + Z) = 0 \\ \frac{dZ}{dV} = Z' = \frac{Z^2 - V^2}{Z^2} \end{gathered} \tag{16}$$

Replace $Z = VF(V)$ to get:

$$\begin{gathered} V\frac{dF}{dV} + F = \frac{F^2 - 1}{F^2} \;\rightarrow\; V\frac{dF}{dV} = \frac{F^2 - 1 - F^3}{F^2} \\ \frac{dV}{V} = \frac{F^2}{-F^3 + F^2 - 1}dF \;\rightarrow\; ln(V) = \int \frac{F^2}{-F^3 + F^2 - 1}dF \end{gathered} \tag{17}$$

The denominator $-F^3 + F^2 - 1$ can be decomposed using three roots - two complex and one real:

$$\begin{gathered} \int \frac{F^2}{-(F - r_1)(F - r_2)(F - r_3)}dF = -\frac{r_1^2 \ln|F - r_1|}{(r_1 - r_2)(r_1 - r_3)} \\ -\frac{r_2^2 \ln|F - r_2|}{(r_2 - r_1)(r_2 - r_3)} - \frac{r_3^2 \ln|F - r_3|}{(r_3 - r_1)(r_3 - r_2)} + c \end{gathered} \tag{18}$$

$$V = C\ e^{\left(-\frac{r_1^2 \ln|F-r_1|}{(r_1-r_2)(r_1-r_3)} - \frac{r_2^2 \ln|F-r_2|}{(r_2-r_1)(r_2-r_3)} - \frac{r_3^2 \ln|F-r_3|}{(r_3-r_1)(r_3-r_2)}\right)}$$
$$V = C(|F-r_1|)^{\frac{-r_1^2}{(r_1-r_2)(r_1-r_3)}}(|F-r_2|)^{\frac{-r_2^2}{(r_2-r_1)(r_2-r_3)}}(|F-r_3|)^{\frac{-r_3^2}{(r_3-r_1)(r_3-r_2)}} \tag{19}$$

In the solution above $F \neq r_1$, $F \neq r_2$ and $F \neq r_3$ since it was implicitly assumed that these values are not equal to zero when dividing by the corresponding polynomial. From the parametric expression it is seen that when F approaches either of the roots, the value of the function becomes infinitely large, which may not be a solution to the bidding function. Hence, it is necessary to examine whether the roots of the polynomial are the solution to equation (16). In other words, check if $Z = kV$ solves this equation:

$$k = \frac{(kV)^2 - V^2}{(kV)^2} \Rightarrow k^3 = k^2 - 1$$

The last expression is exactly the equation with the roots found previously. Considering only the real root the solution becomes $k = -0.7549$ or $Z = -0.7549V$.

Substituting back $Z = z - p$ and $V = v - p$ produces $z - p = -\frac{3}{4}(v - p)$. Solving for z provides the final equation for the exit function:

$$z(v) = \frac{7}{4}p - \frac{3}{4}v \tag{20}$$

The solution is defined only for $v > p$. Function $z(v)$ gives each valuation $v \in (p, 1]$ a corresponding exit price c. Note that the function is decreasing. Hence, buyers with higher valuations exit the auction first. It engenders that following this strategy, a buyer has zero probability of obtaining the object outside if there is a buyer with a higher valuation. However, if several buyers exit the auction before it starts, then a buyer has strictly positive probability of obtaining the object outside. This results in a well-documented discontinuity of the cutoff function at the point of an indifferent buyer with valuation $\hat{v}$ (e.g. Mathews and Katzman (2006), Reynolds and Wooders (2009)). Moreover, conditional on the posted price, an indifferent buyer may either exist or not. Let this price be $\tilde{p}$. If $p > \tilde{p}$ then an indifferent buyer does not exist. If $p < \tilde{p}$ then an indifferent buyer exists, and his valuation is equal to $\hat{v}$.

□

B Proof of Lemma 2

Here, the construction of the expected profit is done for the case when an indifferent buyer exists (i.e. for $p \leq \frac{3}{5}$). It is straightforward to adjust it for the case when an indifferent buyer does not exist. For exposition purposes I derive cumulative profit, which can then be decomposed into the profits gained from the auction and the posted price. Recall that x_1 is the first order statistic among buyers' valuations, x_2 - second order statistic and x_3 - third order statistic. There are several possibilities of realization of their valuations.

a) $x_1, x_2, x_3 < p$. We know that the profit of the seller using a standard second-price auction is equal to the second highest valuation. Since bidders cannot afford the good at the posted price, the profit of the other seller using posted price is zero: $\int_0^p \int_0^{x_1} \int_0^{x_2} (x_2 + 0)6dx_3dx_2dx_1$.

b) $p < x_1 < \frac{5}{3}p$ and $x_3 < x_2 < c$. In this case, both second and third order statistics drop out before the auction price reaches the exit price of the first order statistic, and x_1 wins the auction paying the second highest valuation. The good outside remains unsold: $\int_p^{\frac{5}{3}p} \int_0^c \int_0^{x_2} (x_2 + 0)6dx_3dx_2dx_1$.

c) (i) $p < x_1 < \frac{5}{3}p$, $c < x_2 < p$ and $x_3 < c$. In this case, the third order statistic drops out before the auction price reaches the exit price of the first order statistic. Remaining buyers continue bidding in the auction until the second order statistic drops out. His valuation is the revenue of the seller using the auction. The seller with the good outside receives nothing: $\int_p^{\frac{5}{3}p} \int_c^p \int_0^c (x_2 + 0)6dx_3dx_2dx_1$.

(ii) $p < x_1 < \frac{5}{3}p$, $c < x_2 < p$ and $c < x_3 < x_2$. In this case, the first order statistic exits the auction and buys the good at the posted price. The auction is won by the second order statistic who pays the price equal to the valuation of the third order statistic: $\int_p^{\frac{5}{3}p} \int_c^p \int_c^{x_2} (x_3 + p)6dx_3dx_2dx_1$.

d) (i) $p < x_1 < \frac{5}{3}p$, $p < x_2 < x_1$ and $x_3 < c$. In this case, the third order statistic drops out before the auction price reaches the exit price of the first order statistic, and the other buyers continue to bid in the auction according to $max[v_i, p]$. The auction price rises until it matches the posted price outside. One of the remaining bidders wins the auction, the other one — buys the good at the posted price; they both pay the price p: $\int_p^{\frac{5}{3}p} \int_p^{x_1} \int_0^c (p + p)6dx_3dx_2dx_1$.

(ii) $p < x_1 < \frac{5}{3}p$, $p < x_2 < x_1$ and $c < x_3 < p$. In this case, the first order statistic exits the auction at the exit price and buys the good outside. The auction continues with the other buyers bidding up until the valuation of the third order statistic: $\int_p^{\frac{5}{3}p} \int_p^{x_1} \int_c^p (x_3 + p)6dx_3dx_2dx_1$.

(iii) $p < x_1 < \frac{5}{3}p$, $p < x_2 < x_1$ and $p < x_3 < x_2$. This case is identical to the previous case: $\int_p^{\frac{5}{3}p} \int_p^{x_1} \int_p^{x_2} (x_3 + p)6dx_3dx_2dx_1$.

The next region is tricky, because it is the region affected by the employed rationing scheme. Recall that buyers with $v > \hat{v}$ attempt to buy the good at the posted price at the beginning of the auction. Since every buyer is equally likely to obtain the good, the profits of the seller using the auction will change conditional on who of the buyers succeeds:

f) (i) $\frac{5}{3}p < x_1 < 1$, $0 < x_2 < \frac{5}{3}p$ and $0 < x_3 < x_2$. In this case, the first order statistic does not participate in the auction and buys the good at the posted price right away. The auction is carried out between the other two buyers who bid their valuations. The auction ends at the valuation of the third order statistic: $\int_{\frac{5}{3}p}^{1}\int_0^{\frac{5}{3}p}\int_0^{x_2}(x_3+p)6dx_3dx_2dx_1$.

(ii) $\frac{5}{3}p < x_1 < 1$, $\frac{5}{3}p < x_2 < x_1$ and $0 < x_3 < \frac{5}{3}p$. In this case, both first and second order statistics attempt to buy the object outside at the beginning of the auction, and one of them succeeds. The other one returns to the auction and bids against the third order statistic eventually winning the auction and paying the price equivalent to the value of the latter: $\int_{\frac{5}{3}p}^{1}\int_{\frac{5}{3}p}^{x_1}\int_0^{\frac{5}{3}p}(x_3+p)6dx_3dx_2dx_1$.

(iii) $\frac{5}{3}p < x_1 < 1$, $\frac{5}{3}p < x_2 < x_1$ and $\frac{5}{3}p < x_3 < x_2$. When all three buyers' values are above the valuation of the indifferent buyer, they all attempt to buy the object outside at the beginning of the auction. Since every buyer has equal probability of obtaining the object, in 1/3 of the cases, the buyer with the lowest value (third order statistic) gets the object. The remaining buyers return to the auction and bid against each other. The auction ends at the valuation of the second order statistic. It is readily seen that in 2/3 the of cases, either x_1 or x_2 gets the object. Hence, the auction ends at the valuation of the third order statistic: $\int_{\frac{5}{3}p}^{1}\int_{\frac{5}{3}p}^{x_1}\int_{\frac{5}{3}p}^{x_2}(\frac{1}{3}x_2+\frac{2}{3}x_3+p)6dx_3dx_2dx_1$.

When an indifferent buyer does not exist, the last three cases disappear, and the domain of integration for dx_1 with $x_1 > p$ becomes "$\int_p^1$" instead of "$\int_p^{\frac{5}{3}p}$".

Hence, decomposing and summarizing, the cumulative payoff for the case when an indifferent buyer exists produces the following profit functions:

$$\begin{aligned}
\pi_A|p \le \frac{3}{5} = 6\Bigg[&\int_0^p\int_0^{x_1}\int_0^{x_2} x_2dx_3dx_2dx_1 + \int_p^{\frac{5}{3}p}\int_0^{\frac{7}{4}p-\frac{3}{4}x_1}\int_0^{x_2} x_2dx_3dx_2dx_1 \\
&+\int_p^{\frac{5}{3}p}\int_{\frac{7}{4}p-\frac{3}{4}x_1}^{p}\int_0^{\frac{7}{4}p-\frac{3}{4}x_1} x_2dx_3dx_2dx_1 + \int_p^{\frac{5}{3}p}\int_{\frac{7}{4}p-\frac{3}{4}x_1}^{p}\int_{\frac{7}{4}p-\frac{3}{4}x_1}^{x_2} x_3dx_3dx_2dx_1 \\
&+\int_p^{\frac{5}{3}p}\int_p^{x_1}\int_0^{\frac{7}{4}p-\frac{3}{4}x_1} pdx_3dx_2dx_1 + \int_p^{\frac{5}{3}p}\int_p^{x_1}\int_{\frac{7}{4}p-\frac{3}{4}x_1}^{p} x_3dx_3dx_2dx_1 \\
&+\int_p^{\frac{5}{3}p}\int_p^{x_1}\int_p^{x_2} x_3dx_3dx_2dx_1 + \int_{\frac{5}{3}p}^{1}\int_0^{\frac{5}{3}p}\int_0^{x_2} x_3dx_3dx_2dx_1 \\
&+\int_{\frac{5}{3}p}^{1}\int_{\frac{5}{3}p}^{x_1}\int_0^{\frac{5}{3}p} x_3dx_3dx_2dx_1 + \int_{\frac{5}{3}p}^{1}\int_{\frac{5}{3}p}^{x_1}\int_{\frac{5}{3}p}^{x_3}(\frac{1}{3}x_2+\frac{2}{3}x_3)dx_3dx_2dx_1\Bigg] \\
= 6\Bigg[&\frac{p^4}{12}+\frac{5p^4}{48}+\frac{3p^4}{32}+\frac{p^4}{48}+\frac{4p^4}{27}+\frac{13p^4}{216}+\frac{14p^4}{243}+\Big(\frac{125p^3}{162}-\frac{625p^4}{486}\Big) \\
&+\Big(\frac{25p^2}{36}-\frac{125^3}{54}+\frac{625^4}{324}\Big)+\Big(\frac{1}{18}-\frac{5p}{54}-\frac{25p^2}{54}+\frac{625p^3}{486}-\frac{625p^4}{729}\Big)\Bigg]
\end{aligned}$$

$$= \frac{1}{3} - \frac{5p}{9} + \frac{25p^2}{18} - \frac{125p^3}{81} + \frac{8251p^4}{3888}$$

$$\begin{aligned}
\pi_P | p \leq \frac{3}{5} = 6\Bigg[&\int_p^{\frac{5}{3}p} \int_{\frac{7}{4}p-\frac{3}{4}x_1}^{p} \int_{\frac{7}{4}p-\frac{3}{4}x_1}^{x_2} p dx_3 dx_2 dx_1 \\
&+ \int_p^{\frac{5}{3}p} \int_p^{x_1} \int_0^{\frac{7}{4}p-\frac{3}{4}x_1} p dx_3 dx_2 dx_1 + \int_p^{\frac{5}{3}p} \int_p^{x_1} \int_{\frac{7}{4}p-\frac{3}{4}x_1}^{p} p dx_3 dx_2 dx_1 \\
&+ \int_p^{\frac{5}{3}p} \int_p^{x_1} \int_p^{x_2} p dx_3 dx_2 dx_1 + \int_{\frac{5}{3}p}^{1} \int_0^{x_1} \int_0^{x_2} p dx_3 dx_2 dx_1 \Bigg] \\
&= 6\left[\frac{p^4}{36} + \frac{4p^4}{27} + \frac{2p^4}{27} + \frac{4p^4}{81} + (\frac{p}{6} - \frac{125p^4}{162})\right] = p - \frac{17p^4}{6}
\end{aligned}$$

When an indifferent buyer does not exist, the same procedure results in:

$$\begin{aligned}
\pi_A | p > \frac{3}{5} = 6\Bigg[&\int_0^{p} \int_0^{x_1} \int_0^{x_2} x_2 dx_3 dx_2 dx_1 + \int_p^{1} \int_0^{\frac{7}{4}p-\frac{3}{4}x_1} \int_0^{x_2} x_2 dx_3 dx_2 dx_1 \\
&+ \int_p^{1} \int_{\frac{7}{4}p-\frac{3}{4}x_1}^{p} \int_0^{\frac{7}{4}p-\frac{3}{4}x_1} x_2 dx_3 dx_2 dx_1 + \int_p^{1} \int_{\frac{7}{4}p-\frac{3}{4}x_1}^{p} \int_{\frac{7}{4}p-\frac{3}{4}x_1}^{x_2} x_3 dx_3 dx_2 dx_1 \\
&+ \int_p^{1} \int_p^{x_1} \int_0^{\frac{7}{4}p-\frac{3}{4}x_1} p dx_3 dx_2 dx_1 + \int_p^{1} \int_p^{x_1} \int_{\frac{7}{4}p-\frac{3}{4}x_1}^{p} x_3 dx_3 dx_2 dx_1 \\
&+ \int_p^{1} \int_p^{x_1} \int_p^{x_2} x_3 dx_3 dx_2 dx_1 \Bigg] = -\frac{71}{256} + \frac{135p}{64} - \frac{213p^2}{128} + \frac{7p^3}{64} + \frac{57p^4}{256}
\end{aligned}$$

$$\begin{aligned}
\pi_P | p > \frac{3}{5} = 6\Bigg[&\int_p^{1} \int_{\frac{7}{4}p-\frac{3}{4}x_1}^{p} \int_{\frac{7}{4}p-\frac{3}{4}x_1}^{x_2} p dx_3 dx_2 dx_1 \\
&+ \int_p^{1} \int_p^{x_1} \int_0^{\frac{7}{4}p-\frac{3}{4}x_1} p dx_3 dx_2 dx_1 + \int_p^{1} \int_p^{x_1} \int_{\frac{7}{4}p-\frac{3}{4}x_1}^{p} p dx_3 dx_2 dx_1 \\
&+ \int_p^{1} \int_p^{x_1} \int_p^{x_2} p dx_3 dx_2 dx_1 \Bigg] = \frac{25p}{16} - \frac{27p^2}{16} - \frac{21p^3}{16} + \frac{23p^4}{16}
\end{aligned}$$